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Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves

PETER D. LAX
Courant Institute of Mathematical Sciences
New York University

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Courant Institute of Mathematical Sciences
New York University

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Preface

The mathematical theory of hyperbolic systems of conservation laws and the theory of shock waves presented in these lectures were started by Eberhardt Hopf in 1950, followed in a series of studies by Olga Oleinik, the author, and many others. In 1965, James Glimm introduced a number of strikingly new ideas, the possibilities of which are explored.

In addition to the mathematical work reported here there is a great deal of engineering lore about shock waves; much of that literature up to 1948 is reported in *Supersonic Flow and Shock Waves* by Courant and Friedrichs. Subsequent work, especially in the sixties, relies on a great deal of computation.

A series of lectures, along the lines of these notes, was delivered at a Regional Conference held at the University of California at Los Angeles in September, 1971, arranged by the Conference Board of Mathematical Sciences, and sponsored by the National Science Foundation. The notes themselves are based on lectures delivered at Oregon State University in the summer of 1970, and at Stanford University, summer of 1971. To all these institutions, my thanks, and my thanks also to the Atomic Energy Commission, for its generous support over a number of years of my research on hyperbolic conservation laws. I express my gratitude to Julian Cole and Victor Barcilon, organizers of the regional conference, for bringing together a very stimulating group of people.

New York PETER D. LAX

December 1972

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Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves

Peter D. Lax

Introduction. It is well known that an initial value problem for a nonlinear ordinary differential equation may very well fail to have a solution for all time; the solution may blow up after a finite time. The same is true for quasi-linear hyperbolic partial differential equations: solutions may break down after a finite time when their first derivatives blow up.

In these notes we study first order quasi-linear hyperbolic systems which come from conservation laws. Since a conservation law is an integral relation, it may be satisfied by functions which are not differentiable, not even continuous, merely measurable and bounded. We shall call these *generalized* solutions, in contrast to the regular, i.e., differentiable ones. The breakdown of a regular solution may merely mean that although a generalized solution exists for all time, it ceases to be differentiable after a finite time. All available evidence indicates that this is so. It turns out however that there are many generalized solutions with the same initial data, only one of which has physical significance; the task is to give a criterion for selecting the right one. A class of such criteria is described in these notes; they are called *entropy conditions*, for in the gas dynamical case they amount to requiring the increase of entropy of particles crossing a shock front.

These lectures deal with the mathematical side of the theory, i.e., with results that can be proved rigorously. We present whatever is known about existence and uniqueness of generalized solutions of the initial value problem subject to the entropy conditions. We also investigate the subtle dissipation introduced by the entropy condition and show that it causes a slow decay in signal strength.

As stated in the preface, no numerical results are presented; yet there is a very brief introduction to numerical methods in § 7 of the Notes. The approximate solutions that can be computed by these methods are not only enormously useful quantitatively, but there is hope that such methods can also be used to prove the existence of solutions and to study them qualitatively.

1. Quasi-linear hyperbolic equations. A first order system of quasi-linear equations in two independent variables is of the form

$$u_t + Au_x = 0, \tag{1.1}$$

where u is a vector function of x and t, A a matrix function of u as well as of x and t. Such a system is called strictly hyperbolic if for each x, t and u the matrix $A = A(x, t, u)$ has real and distinct eigenvalues $\tau_j = \tau_j(x, t, u)$, $j = 1, \cdots, n$.

Similarly, a quasi-linear system in $k + 1$ variables $x^1, x^2, \cdots, x^k, t$,

$$u_t + \sum_{i=1}^{k} A_i u_{x^i} = 0, \tag{1.2}$$

$$A_i = A_i(x, t, u),$$

is called hyperbolic if for each x, t, u and unit vector ω, the matrix

$$\sum A_i \omega_i$$

has real and distinct eigenvalues $\tau_j(x, t, u, \omega)$, $j = 1, \cdots, n$.

The initial value problem for (1.1)—or (1.2)—is to find a solution $u(x, t)$ with prescribed values at $t = 0$:

$$u(x, 0) = u_0(x). \tag{1.3}$$

We shall deduce now easily from the linear theory that this initial value problem has at most one solution in the class of C^1 solutions. For let u and v both solve (1.1):

$$u_t + A(u)u_x = 0, \qquad v_t + A(v)v_x = 0,$$

$$u(x, 0) = v(x, 0).$$

Subtracting the two equations we find that the difference $d = u - v$ satisfies

$$d_t + A(u)d_x + [A(u) - A(v)]v_x = 0. \tag{1.4}$$

Assume that A is C^1; then $|A(u) - A(v)| \leqq \text{const.}\, |d|$ for the quasi-linear equation (1.1), and similarly for (1.2), It follows that $d = 0$.

Does the initial value problem always have a solution? We shall sketch an argument, based on linear theory, that the answer is "yes" if the initial values are smooth enough. The solution is obtained by iterating the transformation $u = \mathscr{T} v$ defined as follows: u is the solution of the linear initial value problem

$$u_t + \sum_{i=1}^{k} A_i(v) u_{x^i} = 0,$$

$$u(x, 0) = u_0(x).$$

Let us assume that the A_i are symmetric matrices. It is not hard to show, using the energy estimates for linear symmetric hyperbolic systems and the Sobolev inequalities, that the transformation $\mathscr{T}$ has the following properties:

Suppose that u_0 is of class C^N, where $N > 1 + k/2$. Define the norm $\|u\|_{N,T}$ by

$$\|u\|_{N,T}^2 = \sup_{0 \leqq t \leqq T} \int \sum_{|\alpha| \leqq N} |D_x^\alpha u|^2 \, dx.$$

(i) For R large enough and T small enough $\mathcal{T}$ maps the ball $B_{N,T}^R$ of radius R:

$$\|v\|_{N,T} \leqq R$$

into itself.

(ii) $\mathcal{T}$ is a contraction of $B_{N,T}^R$ with respect to the $\|\cdot\|_{0,T}$ norm:

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|_{0,T} \leqq \tfrac{1}{2}\|v_1 - v_2\|_{0,T}.$$

Since $B_{N,T}^R$ is closed in the $\|\cdot\|_{0,T}$ norm, it follows that $\mathcal{T}$ has a unique fixed point in $B_{N,T}^R$ constructible by iteration. This fixed point solves the initial value problem for the quasi-linear equation (1.2).

This proof shows that (1.2), (1.3) has a solution in some finite time interval $0 \leqq t \leqq T$. Examples presented in § 3 and § 6 show that, in general, smooth solutions do not exist beyond some finite time interval. Since solutions are supposed to describe the state of a physical system, how is one to interpret the nonexistence of solutions in the large? We shall show in the next section that for quasi-linear equations which come from conservation laws there is a way of defining generalized solutions.

2. Conservation laws. A conservation law asserts that the rate of change of the *total amount* of substance contained in a fixed domain G is equal to the *flux* of that substance across the boundary of G. Denoting the *density* of that substance by u, and the flux by f, the conservation law is

$$\frac{d}{dt}\int_G u\,dx = -\int_{\partial G} f\cdot n\,dS; \tag{2.1}$$

here n denotes the outward normal to G and dS the surface element on ∂G, the boundary of G, so that the integral on the right in (2.1) measures outflow—hence the minus sign. Applying the divergence theorem and taking d/dt under the integral sign we obtain

$$\int_G (u_t + \operatorname{div} f)\,dx = 0.$$

Dividing by vol(G) and shrinking G to a point where all partial derivatives of u and f are continuous we obtain the differential conservation law

$$u_t + \operatorname{div} f = 0. \tag{2.2}$$

We shall deal with systems of conservation laws

$$u_t^j + \operatorname{div} f^j = 0, \qquad j = 1, \cdots, n, \tag{2.3}$$

where each f^j is some nonlinear function of $u^1, \cdots, u^n$. Carrying out the differentiations in (2.3) we get the first order quasi-linear system

$$u_t^j + \sum \frac{\partial f_i^j}{\partial u^k} u_{x^i}^k = 0. \tag{2.4}$$

Introducing the vector and matrix notation

$$u = \begin{pmatrix} u^1 \\ \cdot \\ \cdot \\ u^n \end{pmatrix}, \quad f_i = \begin{pmatrix} f_i^1 \\ \cdot \\ \cdot \\ f_i^n \end{pmatrix}, \tag{2.5}$$

$$A_i = \left(\frac{\partial f_i^j}{\partial u^k}\right) = \operatorname{grad}_u f_i,$$

we can write (2.4) in the form

$$u_t + \sum A_i u_{x^i} = 0. \tag{2.6}$$

Since the f^j are nonlinear functions of u, the matrices A_i as defined by (2.5) are functions of u. We assume that the quasi-linear system (2.6) is strictly hyperbolic.

u is called a generalized solution of the system of conservation laws (2.3) *if it satisfies the integral form of these laws, i.e., if*

$$\int_G u^j \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\partial G} f^j \cdot n \, dS \, dt = 0 \tag{2.7}$$

holds for every smoothly bounded domain and for every time interval (t_1, t_2). This is equivalent to requiring (2.3) *to hold in the sense of distribution theory.*

Let $S(t)$ be a smooth surface moving with t, u a continuously differentiable solution of (2.3) on either side of S which is discontinuous across S; the condition which must be satisfied at each point of S if u is a generalized solution across S is

$$s[u^j] = [f^j] \cdot n. \tag{2.8}$$

Here $[u]$ and $[f]$ denote the difference between values of u and f respectively on the two sides of S; n is the normal to S and s the speed with which S propagates in the direction n. Relation (2.8) is called the Rankine–Hugoniot jump condition; we shall prove it for the one-dimensional case in § 3.

We leave now these formal considerations and turn to solving the initial value problem within the class of these generalized solutions.

3. Single conservation laws. A single *conservation law* is an equation of the form

$$u_t + f_x = 0, \tag{3.1}$$

where f is some nonlinear function of u. Denoting

$$\frac{df}{du} = a(u) \tag{3.2}$$

we can write (3.1) in the form

$$u_t + a(u)u_x = 0, \tag{3.3}$$

which asserts that u is constant along trajectories $x = x(t)$ which propagate with speed a:

$$\frac{dx}{dt} = a. \tag{3.4}$$

For this reason a is called the *signal speed*, the trajectories, satisfying (3.4), are called *characteristics*. Note that if f is a nonlinear function of u, both signal speed and characteristic depend on the solution u.

The constancy of u along characteristics combined with (3.4) shows that the characteristics propagate with constant speed; so they are straight lines. This leads to the following geometric solution of the initial value problem $u(x, 0) = u_0(x)$: Draw straight lines issuing from points y of the x-axis, with speed $u_0(y)$ (see Fig. 1).

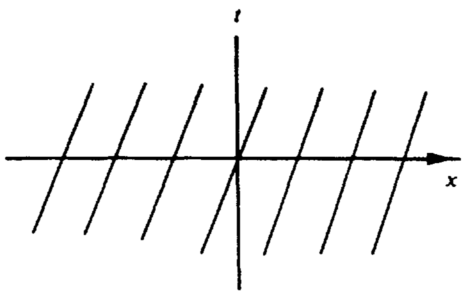

FIG. 1

As we shall show, if u_0 is a C^1 function, these straight lines simply cover a neighborhood of the x-axis. Since the value of u along the line issuing from the point y is $u_0(y)$, $u(x, t)$ is uniquely determined near the x-axis.

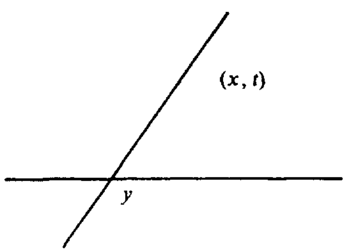

FIG. 2

An analytical form of this construction is shown in Fig. 2. Let (x, t) be any point, y the intersection of the characteristic through x, t with the x-axis. Then $u = u(x, t)$ satisfies

$$u - u_0(x - ta(u)) = 0. \tag{3.5}$$

Assume u_0 differentiable; then, according to the implicit function theorem, (3.5) can be solved for u as a differentiable function of x and t for t small enough, and

$$u_t = -\frac{u_0' a}{1 + u_0' a_u t},$$
$$u_x = \frac{u_0'}{1 + u_0' a_u t}. \tag{3.6}$$

Substituting (3.6) into (3.3) we see immediately that u defined by (3.5) satisfies (3.3).

Let us assume that (3.3) is genuinely nonlinear, i.e., that $a_u \neq 0$ for all u, say

$$a_u > 0. \tag{3.7}$$

Then if u_0' is $\geqq 0$ for all x, u_t and u_x as given by formulas (3.6) remain bounded for all $t > 0$; on the other hand, if u_0' is < 0 at some point, both u_t and u_x tend to ∞ as $1 + u_0' a_u(u_0)t$ approaches zero. Both these facts can be deduced from the geometric form of the solution contained in Fig. 1 as follows.

In the first case, when $u_0(x)$ is an increasing function of x, the characteristics issuing from the x-axis diverge in the positive t direction, so that the characteristics simply cover the whole half-plane $t > 0$. In the second case there are two points y_1 and y_2 such that $y_1 < y_2$, and $u_1 = u_0(y_1) > u_0(y_2) = u_2$; then by (3.7) also $a_1 = a(u_1) > a(u_2) = a_2$ so that the characteristics issuing from these points intersect at time $t = (y_2 - y_1)/(a_1 - a_2)$ (see Fig. 3). At the point of intersection, u has to take on both values u_1 and u_2, an impossibility.

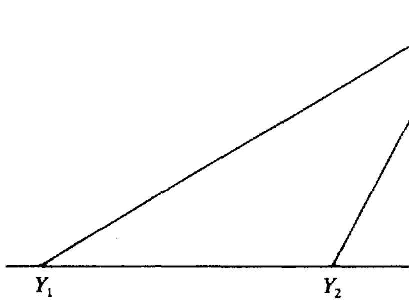

FIG. 3

Both the geometric and the analytic argument prove beyond the shadow of a doubt that if $a(u_0(x))$ is not an increasing function of x, then *no function $u(x, t)$ exists for all $t > 0$ with initial value u_0 which solves* (3.3) *in the ordinary sense*! We saw however in § 2 that bounded, measurable functions u which satisfy (3.1) in the sense of distributions can be regarded as satisfying the integral form of the conservation law of which (3.1) is the differential form. We turn now to the study of such distribution solutions, starting with the simplest kind—those satisfying

(3.1) in the ordinary sense on each side of a smooth curve $x = y(t)$ across which u is discontinuous. We shall denote by u_l and u_r the values of u on the left and right sides, respectively, of $x = y$. Choose a and b so that the curve y intersects the

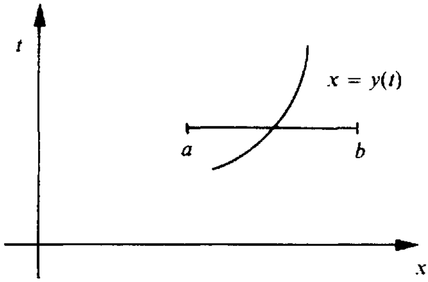

FIG. 4

interval $a \leqq x \leqq b$ at time t (see Fig. 4). Denoting by $I(t)$ the quantity

$$I(t) = \int_a^b u(x, t)\,dx = \int_a^y + \int_y^b,$$

we have

$$\frac{dI}{dt} = \int_a^y u_t\,dx + u_l s + \int_y^b u_t\,dx - u_r s, \tag{3.8}$$

where we have used the abbreviation

$$s = \frac{dy}{dt} \tag{3.9}$$

for the speed with which the discontinuity propagates. Since on either side of the discontinuity (3.1) is satisfied, we may set $u_t = -f_x$ in (3.8) for $x < y$ and $x > y$, obtaining after carrying out the integration that

$$\frac{dI}{dt} = f_a - f_l + u_l s - f_b + f_r - u_r s.$$

Here we have used the handy abbreviations

$$f(u_l) = f_l, \qquad f(u_r) = f_r,$$
$$f(u(a)) = f_a, \qquad f(u(b)) = f_b.$$

The conservation law asserts that

$$\frac{dI}{dt} = f_a - f_b.$$

Combining this with the above relation we deduce the *jump condition*

$$s[u] = [f], \tag{3.10}$$

where $[u] = u_r - u_l$ and $[f] = f_r - f_l$ denote the jump in u and in f across y.

We show now in an example that previously unsolvable initial value problems can be solved for all t with the aid of discontinuous solutions. Take

$$f(u) = \tfrac{1}{2}u^2, \tag{3.11}$$

$$u_0(x) = \begin{cases} 1 & \text{for} \quad x \leqq 0, \\ 1 - x & \text{for} \quad 0 \leqq x \leqq 1, \\ 0 & \text{for} \quad 1 \leqq x. \end{cases}$$

The geometric solution (see Fig. 5) is single-valued for $t \leqq 1$ but double-valued

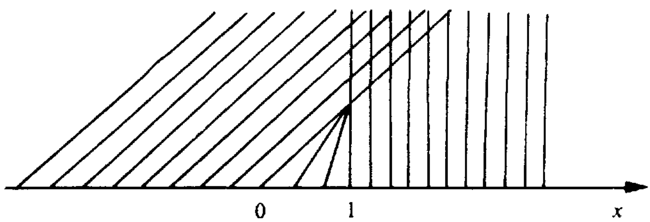

FIG. 5

thereafter. Now we define for $t \geqq 1$,

$$u(x, t) = \begin{cases} 1 & \text{for} \quad x < \dfrac{1 + t}{2}, \\ 0 & \text{for} \quad \dfrac{1+t}{2} < x. \end{cases}$$

The discontinuity starts at (1.1); it separates the state $u_l = 1$ on the left from the state $u_r = 0$ on the right; the speed of propagation was chosen according to the jump condition (3.10), with $f(u) = \frac{1}{2}u^2$:

$$s = \frac{0 - 1/2}{0 - 1} = \frac{1}{2}.$$

Introducing generalized solutions makes it possible to solve initial value problems which could not be solved within the class of genuine solutions. At the same time it threatens with the danger that the enlarged class of solutions is so large that there are several generalized solutions with the same initial data. The following example shows that this anxiety is well founded:

$$u_0(x) = \begin{cases} 0 & \text{for} \quad x < 0, \\ 1 & \text{for} \quad 0 < x. \end{cases}$$

The geometric solution (see Fig. 6) is single-valued for $t > 0$ but does not deter-

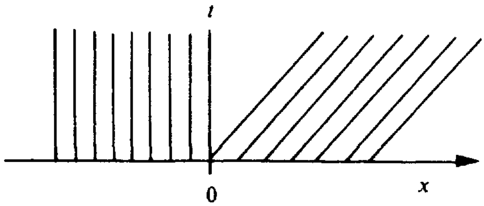

FIG. 6

mine the value of u in the wedge $0 < x < t$. We could fill this gap in the fashion of the previous example and set

$$u(x,t) = \begin{cases} 0 & \text{for} \quad x < \dfrac{t}{2}, \\ 1 & \text{for} \quad \dfrac{t}{2} < x. \end{cases} \tag{3.12}$$

The speed of propagation was so chosen that the jump condition (3.10) is satisfied. On the other hand, the function

$$u(x,t) = \frac{x}{t}, \qquad 0 \leqq x \leqq t, \tag{3.12'}$$

satisfies the differential equation (3.3) with $a(u) = u$, and joins continuously the rest of the solution determined geometrically. Clearly only one of these solutions can have physical meaning; the question is which?

We reject the discontinuous solution (3.12) for failure to satisfy the following criterion:

The characteristics starting on either side of the discontinuity curve when continued in the direction of increasing t intersect the line of discontinuity. This will be the case if

$$a(u_l) > s > a(u_r). \tag{3.13}$$

Clearly this condition is violated in the solution given by (3.12).

If all discontinuities of a generalized solution satisfy condition (3.13), no characteristic drawn in the direction of decreasing t intersects a line of discontinuity. This shows that for such solutions every point can be connected by a backward drawn characteristic to a point on the initial line; therein lies the significance of condition (3.13). When applied to the equations of compressible flow, this generalization amounts to requiring that material which crosses the discontinuity should suffer an increase of entropy. For this reason condition (3.13) will be called the *entropy condition.*

A discontinuity satisfying the jump relation (3.10) and the entropy condition (3.13) is called a *shock.* The task before us is to investigate whether every initial value problem for (3.1) has exactly one generalized solution, defined for all $t \geqq 0$, which has only shocks as discontinuities.

We shall first treat the case when condition (3.7) is satisfied, i.e., $a(u)$ is an increasing function of u. Clearly this is so whenever $f(u)$ is a *convex* function of u

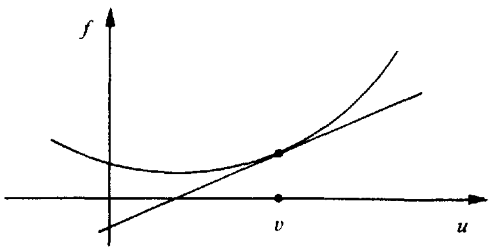

FIG. 7

(see Fig. 7). Such a function lies above all its tangents:

$$f(u) \geqq f(v) + a(v)(u - v). \tag{3.14}$$

Let u be a genuine (i.e., continuous and differentiable) solution of (3.1), and suppose that $u_0(x) = u(x, 0)$ is 0 for x large enough negative; then the same is true of $u(x, t)$ for any $t > 0$ for which u is defined. We introduce the integrated function $U(x, t)$ defined as follows:

$$U(x, t) = \int_{-\infty}^{x} u(y, t)\, dy; \tag{3.15}$$

then

$$U_x = u. \tag{3.16}$$

Integrating (3.1) from $-\infty$ to x and using (3.16) we obtain

$$U_t + f(U_x) = 0, \tag{3.17}$$

where we have adjusted f so that

$$f(0) = 0. \tag{3.18}$$

Applying inequality (3.14) with $U_x = u$ and any number v we obtain that

$$U_t + a(v)U_x \leqq a(v)v - f(v). \tag{3.19}$$

Denote by y the point where the line $dx/dt = a(v)$ through x, t intersects the x-axis; clearly,

$$\frac{x - y}{t} = a(v). \tag{3.20}$$

Integrating (3.19) along this line from 0 to t we obtain, for $t \geqq 0$,

$$U(x, t) \leqq U(y, 0) + t[a(v)v - f(v)]. \tag{3.21}$$

Denote by b the inverse of the function a; from (3.20) we obtain

$$b\left(\frac{x-y}{t}\right) = v. \tag{3.22}$$

Denote by g the function

$$g(z) = a(v)v - f(v), \qquad v = b(z). \tag{3.23}$$

Clearly, since a and b are inverse functions,

$$\frac{dg}{dz} = v\frac{da}{dv}\frac{db}{dz} = b(z).$$

Denote $a(0)$ by c; then $b(c) = 0$ and in view of the normalization (3.18), we have from (3.23) that $g(c) = 0$. Introducing the function g on the right side of (3.21) we obtain

$$U(x,t) \leqq U(y,0) + tg\left(\frac{x-y}{t}\right). \tag{3.24}$$

This inequality holds for all choices of y; for that value of y for which v, given by (3.22), equals $u(x, t)$, the sign of equality holds in (3.19) along the whole characteristic $dx/dt = u$ issuing from (x, t); therefore equality also holds in (3.24). We summarize this result as follows.

THEOREM 3.1. *Let u be a genuine solution of* (3.1); *then*

$$u(x,t) = b\left(\frac{x-y}{t}\right), \tag{3.25}$$

where $y = y(x, t)$ is that value which minimizes

$$U_0(y) + tg\left(\frac{x-y}{t}\right) = G(x,y,t). \tag{3.26}$$

Here b is the inverse function of a, and g is defined by

$$\frac{dg}{dz} = b(z), \qquad g(c) = 0, \quad \textit{where} \quad a(0) = c, \tag{3.27}$$

and

$$U_0(y) = \int_{-\infty}^{y} u_0(x)\,dx, \qquad u_0(x) = u(x,0). \tag{3.28}$$

Suppose $u(x, t)$ is a generalized solution. Relation (3.17) is the integral form of the conservation law (3.1), so it follows that when f is convex, inequality (3.24) is valid for generalized solutions as well.

If all discontinuities of the generalized solution u are shocks, then every point (x, t) can be connected to a point y on the initial line by a backward characteristic.

For that value of y the sign of equality holds in (3.24); thus Theorem 3.1 applies also to generalized solutions of (3.1) whose discontinuities are shocks. We show that also the converse holds, thereby proving the existence of solutions, with shocks, with arbitrarily prescribed integrable initial data.

THEOREM 3.2. *Formula* (3.25)–(3.26) *defines a possibly discontinuous function* $u(x, t)$ *for arbitrary integrable initial values* $u_0(x)$; *the function* u *so defined satisfies* (3.1) *in the sense of distributions, and the discontinuties of* u *are shocks.*

Proof. (i) If u_0 is integrable, U_0 is bounded. As may be seen from (3.27), g is a convex function which achieves its minimum at $z = c$; therefore, the function G defined by (3.26) achieves its minimum in y at some point or points.

LEMMA 3.3. *For fixed* t, *denote by* $y(x)$ *any value of* y *where* $G(x, y)$ *achieves its minimum. Then* $y(x)$ *is a nondecreasing function of* x.

Proof. We have to show that for $x_2 > x_1$, $G(x_2, y)$ does not take on its minimum for $y < y_1 = y(x_1)$. In particular, we shall show that for $y < y_1$,

$$G(x_2, y_1) < G(x_2, y). \tag{3.29}$$

For by the minimizing property of y_1,

$$G(x_1, y_1) \leqq G(x_1, y). \tag{3.30}$$

From Jensen's inequality we have that for $x_1 < x_2$ and $y < y_1$,

$$g\left(\frac{x_2 - y_1}{t}\right) + g\left(\frac{x_1 - y}{t}\right) < g\left(\frac{x_1 - y_1}{t}\right) + g\left(\frac{x_2 - y}{t}\right).$$

Multiplying this last inequality by t and adding to (3.30) we obtain (3.29); this completes the proof of the lemma.

It follows from the lemma that, for fixed t, for all but a denumerable set of x the minimum of $G(x, y, t)$ is achieved at exactly one point $y = y(x, t)$, which is a monotonic increasing function of x. Thus $u(x, t)$ is well defined by (3.25) at all but these points.

(ii) We can combine (3.25) and (3.26) into one formula

$$u = \lim u_N,$$

where

$$u_N(x, t) = \frac{\int b[(x - y)/t] \exp\{-NG\}\, dy}{\int \exp\{-NG\}\, dy}.$$

Similarly,

$$f(u) = \lim f_N,$$

where

$$f_N(x, t) = \frac{\int f(b((x - y)/t)) \exp\{-NG\}\, dy}{\int \exp\{-NG\}\, dy}.$$

Denote by V_N the function

$$V_N = \log \int \exp\{-NG\}\, dy.$$

Clearly,

$$u_N = -\frac{1}{N}\frac{\partial}{\partial x}V_N$$

and similarly we see that

$$f_N = \frac{1}{N}\frac{\partial}{\partial t}V_N$$

provided that we make use of the identity

$$zb(z) - g(z) = f(b(z)).$$

It follows from these relations that

$$u_{N_t} + f_{N_x} = 0;$$

letting $N \to \infty$ we obtain in the limit relation (3.1) for u.

(iii) Since Lemma 3.3, $y(x, t)$ is an increasing function of x, and since b is an increasing function, we have for $x_1 < x_2$ that

$$\begin{aligned} u(x_1, t) &= b\left(\frac{x_1 - y_1}{t}\right) \geqq b\left(\frac{x_1 - y_2}{t}\right) \\ &\geqq b\left(\frac{x_2 - y_2}{t}\right) - k\frac{x_2 - x_1}{t} \\ &= u(x_2, t) - k\frac{x_2 - x_1}{t}; \end{aligned}$$

here k is a Lipschitz constant for b. The result is a one-sided Lipschitz condition for u:

$$\frac{u(x_2, t) - u(x_1, t)}{x_2 - x_1} \leqq \frac{k}{t}, \tag{3.31}$$

which shows that at points of discontinuity $u_r < u_l$, as required by the shock condition.

(iv) The solutions defined by (3.25), (3.26) have the semigroup property, i.e., that if $u(x, t_1)$ is taken as new initial value, the corresponding solution at time t_2 furnished by (3.25), (3.26) equals $u(x, t_1 + t_2)$. This is easily verified by a simple argument.

We turn now to the problem of uniqueness.

THEOREM 3.4. *Let u and v be two piecewise continuous generalized solutions of* (3.1); *assume that f is convex and that all discontinuities of both u and v are shocks.*

Then

$$\|u(t) - v(t)\|$$

is a decreasing function of t, *where the norm is the* L_1 *norm with respect to the* x *variable.*

COROLLARY. *If* $u = v$ *at* $t = 0$, $u = v$ *for all* $t > 0$. (This is the *uniqueness theorem* we were looking for.)

Proof. We can write the L_1 norm of $u - v$ as

$$\|u - v\| = \sum(-1)^n \int_{y_n}^{y_{n+1}} (u - v)\, dx, \tag{3.32}$$

where the points are so chosen that

$$u(x, t) - v(x, t) \quad \text{has the sign} \quad (-1)^n$$

for $y_n < x < y_{n+1}$; of course the y_n are functions of t.

There are two cases:

(i) y_n is a point of continuity of both u and v; in this case

$$u(y_n, t) = v(y_n, t). \tag{3.33}$$

Since values of solutions are constant along characteristics, it follows that in this case y_n is a linear function of t.

(ii) y_n is a point of discontinuity of one of the functions u or v; in this case y_n is a shock curve.

Differentiate (3.32) with respect to t:

$$\begin{aligned} \frac{d}{dt}\|u - v\| = \sum(-1)^n \Bigg[&\int_{y_n}^{y_{n+1}} \frac{\partial}{\partial t}(u - v)\, dx \\ &+ (u - v)(y_{n+1})\frac{dy_{n+1}}{dt} - (u - v)(y_n)\frac{dy_n}{dt}\Bigg]. \end{aligned} \tag{3.34}$$

On the interval y_n, y_{n+1} both u and v satisfy (3.1) in the sense of distributions. Setting $u_t = -f(u)_x$, $v_t = -f(v)_x$ in (3.34) we obtain, after carrying out the integration,

$$(-1)^n(f(v) - f(u))(y) + (u - v)(y)\frac{dy}{dt}\Bigg|_{y_n}^{y_{n+1}}. \tag{3.35}$$

In case (i) $u(y) = v(y)$ which makes (3.35) equal 0. We turn now to case (ii). Suppose that u has a shock at $y = y_{n+1}$ and that $v = v(g)$ lies between u_l and u_r:

$$u_r < v < u_l. \tag{3.36}$$

Then $u - v$ is positive in (y_n, y_{n+1}), so n is even. According to the jump relation (3.10),

$$\frac{dy}{dt} = s = \frac{f(u_l) - f(u_r)}{u_l - u_r}.$$

Substituting this into (3.35), we obtain at the endpoint y_{n+1} with $u = u_l$,

$$
\begin{aligned}
&f(v) - f(u_l) + (u_l - v)\frac{f(u_l) - f(u_r)}{u_l - u_r} \\
(3.37)\qquad &= f(v) - \left[\frac{v - u_r}{u_l - u_r} f(u_l) + \frac{u_l - v}{u_l - u_r} f(u_r)\right].
\end{aligned}
$$

Since f is a convex function and since by (3.36) v lies in the interval (u_r, u_l), it follows from Jensen's inequality that the right side of (3.37) is negative. Similarly, also the contribution at the lower endpoint to (3.35) is negative. This shows that $(d/dt)\|u - v\|$ is always $\leqq 0$, so that as t increases $\|u - v\|$ decreases. This completes the proof of the theorem.

The case when y is a discontinuity for both u and v can be treated similarly.

In the derivation of Theorem 3.4 we can omit the requirement that f be convex if we replace the entropy condition (3.13) by the following:

(i) If $u_r < u_l$, then the graph of f over $[u_r, u_l]$ lies below the chord (see Fig. 8):

$$(3.38_{\text{i}})\qquad f(\alpha u_r + (1 - \alpha)u_l) \leqq \alpha f(u_r) + (1 - \alpha)f(u_l) \quad \text{for} \quad 0 \leqq \alpha \leqq 1.$$

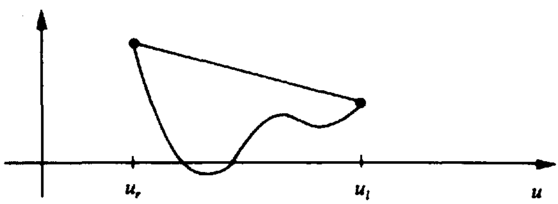

FIG. 8

(ii) If $u_l < u_r$ the graph of f over $[u_l, u_r]$ lies above the chord (see Fig. 9):

$$(3.38_{\text{ii}})\qquad f(\alpha u_r + (1 - \alpha)u_l) \geqq \alpha f(u_r) + (1 - \alpha)f(u_l) \quad \text{for} \quad 0 \leqq \alpha \leqq 1.$$

Conditions (3.38) are called the *generalized entropy conditions.*

Let $f(u)$ be an arbitrary C^1 function of u; let $u(x, t)$ be a distribution solution of (3.1) which is a genuine solution outside of a finite number of discontinuities.

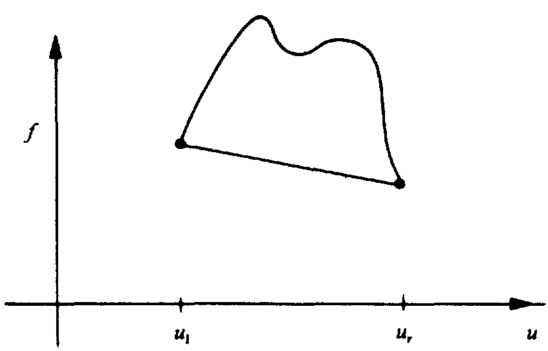

FIG. 9

A discontinuity in u is called a *shock* if u_r and u_l satisfy one of the entropy conditions (3.38).

The proof of Theorem 3.4 yields the following more general result.

THEOREM 3.5. *Let f be any C^1 function, u and v two distribution solutions of* (3.1) *of which all discontinuities are shocks. Then*

$$\|u(t) - v(t)\|$$

is a decreasing function of t.

It follows in particular that *two such solutions which are equal at $t = 0$ are equal for all t.*

The proof of Theorem 3.5 also demonstrates the following converse.

COROLLARY. *If u is a distribution solution of* (3.1) *of which one of the discontinuities fails to satisfy the entropy condition* (3.38), *then there is a genuine solution v such that*

$$\|u(t) - v(t)\|$$

is not a decreasing function of t.

The uniqueness theorem stated above is not very interesting unless we can show that the generalized entropy condition is not too restrictive, i.e., that every initial value problem $u(x, 0) = u_0(x)$ has a distribution solution u of which the discontinuities satisfy the generalized entropy condition. Such a solution u can be constructed, as the limit of solutions u_λ of the parabolic equation

$$\frac{\partial}{\partial t}u_\lambda + \partial_x f(u_\lambda) = \lambda\frac{\partial^2}{\partial x^2}u_\lambda, \qquad \lambda > 0, \tag{3.39}$$

$$u_\lambda(x, 0) = u_0(x).$$

It follows easily from the maximum principle that (3.39) has at most one solution; it is also true that a solution u_λ exists for all t, and that as $\lambda \to 0$ these solutions converge in the L_1 sense to a limit u. We shall not present these proofs, but we shall show that this limit is a distribution solution of (3.1) which satisfies the generalized entropy condition. To show the truth of the first statement we multiply (3.39) by a C_0^∞ test function ϕ and integrate by parts; we obtain

$$-\int\int [\phi_t u_\lambda + \phi_x f(u_\lambda)]\, dx\, dt = \lambda\int \phi_{xx} u_\lambda.$$

Let $\lambda \to 0$; the left side converges to

$$-\int\int \phi_t u + \phi_x f(u)]\, dx\, dt,$$

the right side converges to 0. This proves that u is a distribution solution of (3.1).

To prove that u satisfies the entropy condition, we show first that for any two solutions u_λ and v_λ of (3.39),

$$\|u_\lambda(t) - v_\lambda(t)\|$$

is a decreasing function of t. To see this we write

$$\|u_\lambda - v_\lambda\| = \sum(-1)^n \int_{y_n}^{y_{n+1}} (u_\lambda - v_\lambda)\, dx,$$

where $u_\lambda - v_\lambda$ changes sign at the points y_n. Since solutions of (3.39) are continuous,

$$u_\lambda - v_\lambda = 0 \quad \text{at} \quad y_n. \tag{3.40}$$

Differentiate $\|u_\lambda - v_\lambda\|$:

$$\frac{d}{dt}\|u_\lambda - v_\lambda\| = \sum(-1)^n \int_{y_n}^{y_{n+1}} \frac{\partial}{\partial t}(u_\lambda - v_\lambda)\, dx + \sum(-1)^n (u_\lambda - v_\lambda)\frac{dy}{dt}\bigg|_{y_n}^{y_{n+1}}.$$

On account of (3.40) the second term on the right is zero; substituting from (3.39) into the first term and carrying out the integration we obtain

$$\frac{d}{dt}\|u_\lambda - v_\lambda\| = \sum(-1)^n \lambda \frac{\partial}{\partial x}(u_\lambda - v_\lambda)\bigg|_{y_n}^{y_{n+1}} - \sum(-1)^n (f(u_\lambda) - f(v_\lambda))\bigg|_{y_n}^{y_{n+1}}.$$

Again the second sum on the right is zero, on account of (3.40). We claim that each term in the first sum is nonpositive; for $(-1)^n \lambda(u_\lambda - v_\lambda) = p$ is nonnegative in the interval and by (3.40) zero at the endpoints; so p_x is nonnegative at the left endpoint, nonpositive at the right endpoint. This completes the proof that

$$\|u_\lambda(t) - v_\lambda(t)\|$$

is a decreasing function of t. If u is the L_1 limit of u_λ and v the L_1 limit of v_λ, also $\|u(t) - v(t)\|$ is decreasing.

It is not hard to show that every genuine solution v of (3.1) is the limit as $\lambda \to 0$ of the solutions v_λ of the parabolic equation. Therefore it follows that if u is a distribution solution of (3.1) which is an L_1 limit of solutions u_λ of (3.39) and if v is any genuine solution of (3.1), then $\|u(t) - v(t)\|$ is a decreasing function of t. According to the corollary to Theorem 3.5 it follows then that all discontinuities of u satisfy the generalized entropy condition, as asserted.

Being rid of the convexity conditions makes it possible to extend these notions and the existence and uniqueness theorems to single conservation laws in any number of space variables.

4. The decay of solutions as t tends to infinity. Suppose $f(u)$ is a convex function; then Theorem 3.2 gives an explicit expression for the solutions u of (3.1) in terms of their initial data:

$$u(x, t) = b\left(\frac{x - y}{t}\right), \tag{4.1}$$

where y minimizes

$$G(x, y, t) = U_0(y) + tg\left(\frac{x - y}{t}\right). \tag{4.2}$$

Let us see what we can deduce from (4.1) about the behavior of solutions for large t. We recall that g is a convex function and b a monotonically increasing one, and that g takes on its minimum at $c = a(0)$ and

$$g(c) = 0. \tag{4.3}$$

We denote by k the quantity

$$k = \tfrac{1}{2}b'(c) = \tfrac{1}{2}g''(c). \tag{4.4}$$

Let us assume that f is strictly convex; then $k > 0$. We further assume that b' lies between two positive constants for all u:

$$k_- < \tfrac{1}{2}b' < k_+. \tag{4.5}$$

It follows from this and (4.3) that

$$g(z) \geqq k_-(z - c)^2; \tag{4.6}$$

then

$$tg\left(\frac{x - y}{t}\right) \geqq \frac{k_-}{t}(x - y - ct)^2.$$

Suppose the initial value u_0 of u is in L_1; then for every y, $U_0(y) = \int_{-\infty}^{y} u_0\,dx$ is bounded in absolute value by $\|u_0\| = M$. Using (4.6) we see that for all y,

$$-M + \frac{k}{t}(x - y - ct)^2 \leqq G(x, y, t). \tag{4.7}$$

$G(x, x + ct, t) = U_0(x + ct)$ is $\leqq M$; this shows that $G(x, y, t) \leqq M$ at the minimizing point. Combining this with (4.7) we see that

$$\left|\frac{x - y}{t} - c\right| \leqq \sqrt{\frac{2M}{kt}}. \tag{4.8}$$

It follows from (4.5) and $b(c) = 0$ that

$$|b(z)| < 2k_+|z - c|;$$

combining this with (4.8) we obtain

$$\left|b\left(\frac{x - y}{t}\right)\right| \leqq \frac{\text{const.}}{\sqrt{t}}, \quad \text{const.} = 2k_+\sqrt{\frac{2M}{k_-}}.$$

Thus, by (4.1),

$$|u(x, t)| \leqq \frac{\text{const.}}{\sqrt{t}}. \tag{4.9}$$

Suppose that the initial value $u_0(x)$ is zero outside the interval $(-A, A)$; then $U_0(y)$ is zero for $x < -A$, some constant for $A < x$. According to (4.8), the minimum value of y lies in the interval

$$x - ct - \text{const.}\sqrt{t} \leqq y \leqq x - ct + \text{const.}\sqrt{t}.$$

If $x < ct - \text{const.}\sqrt{t} - A$, y lies in the interval $y < -A$ where the value of $U_0(y)$ is independent of y; therefore the minimum of G is taken on at the point which minimizes $tg((x - y)/t)$; this value is $y = x - ct$. Similarly, for

$$x > ct + \text{const.}\sqrt{t} + A$$

the minimizing value of y is $y = x - ct$. Since $b(c) = 0$, we conclude from (4.1) that $u(x, t) = 0$ for x outside of

$$(ct - \text{const.}\sqrt{t} - A, ct + \text{const.}\sqrt{t} + A).$$

This result, combined with (4.9), can be expressed thus: Every solution u whose initial value is zero outside a finite interval is, at time t, zero outside an interval whose length is $O(\sqrt{t})$; inside that interval u is $O(1/\sqrt{t})$.

A more detailed analysis of the explicit formula yields the following more precise statement about the behavior of solutions for large t.

THEOREM 4.1. *Define the 2-parameter family of functions $v(p, q)$, $p, q \geqq 0$ as follows:*

$$v(x, t; p, q) = \begin{cases} \dfrac{1}{h}\left(\dfrac{x}{t} - c\right) & \text{for} \quad ct - \sqrt{pt} < x < ct + \sqrt{qt}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.10}$$

Let $u(x, t)$ be any solutions with shocks of

$$u_t + f_x(u) = 0, \tag{4.11}$$

where f is convex, $f'(0) = c$, $f''(0) = h$. Then

$$\|u(t) - v(t; p, q)\|$$

tends to 0 *as $t \to \infty$, where $\|\cdot\|$ is the L_1 norm and*

$$\begin{aligned} p &= -2h \min_y \int_{-\infty}^{y} u_0(x)\,dx, \\ q &= 2h \max_y \int_{y}^{\infty} u_0(x)\,dx. \end{aligned} \tag{4.12}$$

We shall not present a proof of this theorem but we shall present a verification of one of its consequences.

We introduce the following abbreviations:

$$\begin{aligned} \min_y \int_{-\infty}^{y} u(x)\,dx &= I_-(u), \\ \max_y \int_{y}^{\infty} u(x)\,dx &= I_+(u). \end{aligned} \tag{4.13}$$

In terms of these (4.12) can be written as

$$p = -2hI_-(u_0),$$

$$q = 2hI_+(u_0).$$

It follows easily from the definition of v in (4.10) that, for any T,

$$\lim_{t\to\infty} \|v(t+T;p,q) - v(t;p,q)\| = 0.$$

It follows then from Theorem 4.1 that $\|u(t+T;p,q) - v(t,p,q)\| \to 0$; applying (4.12) to $u(x, t+T)$ in place of $u(x, t)$ we conclude that for any T

$$p = -2hI_-(u(T)),$$

$$q = 2hI_+(u(T)).$$

In words: *I_- and I_+ are time invariant functionals of solutions.*

We shall present now a direct proof for the invariance of I_-.

Let u be some solution of (4.11), possibly with shocks; denote by $M(t)$:

$$M(t) = I_-(u) = \min_y \int_{-\infty}^{y} u(x,t)\,dx$$

and by $y(t)$ any of the values y where the indicated minimum is taken on. Our aim is to show that M is independent of t.

According to the integral form of the conservation law, for any t_1 and t_2 and any y,

$$\int_{-\infty}^{y} u(x,t_2)\,dx = \int_{-\infty}^{y} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(y,t))\,dt;$$

here we have used the fact that $u(-\infty, t) = 0$ and that $f(0) = 0$. Taking y to be $y(t_1) = y_1$, $y(t_2) = y_2$, respectively, and using the definition of y as minimum, we obtain the inequalities

$$M(t_2) \leqq M(t_1) + \int_{t_1}^{t_2} f(y_1)\,dt,$$

$$M(t_1) \leqq M(t_2) - \int_{t_1}^{t_2} f(y_2)\,dt,$$

which imply that

$$|M(t_2) - M(t_1)| \leqq |t_1 - t_2| F, \tag{4.14}$$

where

$$F = \sup_{t_1 \leqq t \leqq t_2} |f(y_1,t)|, |f(y_2,t)|; \tag{4.15}$$

here $f(y,t)$ abbreviates $f(u(y,t))$.

LEMMA. *$y(t)$, t is a point of continuity of u, and*

$$u(y,t) = 0. \tag{4.16}$$

Proof. By the minimizing property of y we must have

$$u_l = u(y_-, t) \leqq 0, \qquad u_r = u(y_+, t) \geqq 0. \tag{4.17}$$

Since f was assumed convex, the entropy condition is that $u_l > u_r$, so it follows from (4.17) that $y(t)$ cannot be a point of discontinuity, and that $u(y, t) = 0$.

Since the set of minimizing points $y(t)$, t is closed, it follows that in any compact portion of the (x, t)-plane, $y(t)$, t has a positive distance from any shock of strength $|u_r - u_l| = \varepsilon$. From this it follows that as $t_2 \to t_1$, the oscillation of $u(y_1, t)$ and of $u(y_2, t)$ over (t_1, t_2) tends to 0. According to (4.16), $u(y_1, t_1) = u(y_2, t_2) = 0$, so $u(y_1, t)$ and $u(y_2, t)$ tend to 0 uniformly on (t_1, t_2) as $t_2 \to t_1$. Since $f(0) = 0$, it follows that likewise F, the maximum of f over this interval, tends to 0 as $t_2 \to t_1$. But then we deduce from (4.12) that

$$\frac{dM}{dt} = 0,$$

which proves the constancy of M and shows the invariance of the functional I_-. The invariance of I_+ follows similarly; alternatively we observe that the minimum and maximum in (4.13) occur for the same value of y; this implies that

$$I_-(u) + I_+(u) = \int_{-\infty}^{\infty} u(x)\, dx = I_0(u).$$

It is a consequence of the integral form of the conservation law that for solutions u which are zero at $x = \pm\infty$, $I_0(u)$ is independent of t; thus the invariance of I_+ follows from that of I_- and I_0.

The quantity I_0 is a natural invariant built into the conservation law; it is remarkable that there exist other, "unnatural" invariants.

THEOREM 4.2. *Equation* (3.1) *has exactly* 2 *independent invariant functionals continuous in the* L_1 *topology.*

Proof. Let I be any invariant functional continuous in the L_1 topology; by Theorem 4.1,

$$I(u) = \lim_{t \to \infty} I(v(t, p, q)).$$

The value of the right side is determined by the values of p and q, which in turn are determined by $I_-(u)$ and $I_+(u)$. Therefore, the value of $I(u)$ is a function of the values of $I_+(u)$ and $I_-(u)$, as asserted.

Using the explicit formula (4.1) we deduced, in (4.9) and Theorem 4.1, that solutions whose initial values lie in L_1 decay to 0 as $A \to \infty$. We shall present now another method for studying the behavior of solutions as $t \to \infty$, one that does not rely on the explict formula (4.1). With the aid of this method we can show that, as $t \to \infty$, solutions whose initial values are periodic tend uniformly to their mean value $\bar{u}_0$:

$$\bar{u}_0 = \frac{1}{p}\int_0^p u_0(x)\, dx. \tag{4.18}$$

As remarked in § 3, any differentiable solution u of (4.11) is constant along characteristics

$$\frac{dx}{dt} = a(u) = f'(u). \tag{4.19}$$

Let $x_1(t)$ and $x_2(t)$ be a pair of characteristics, $0 \leqq t \leqq T$. Then there is a whole one-parameter family of characteristics connecting the points of the interval $[x_1(0), x_2(0)]$, $t = 0$ with points of the interval $[x_1(T), x_1(T)]$, $t = T$; since u is constant along these characteristics, *$u(x, 0)$ on the first interval and $u(x, T)$ on the second interval are equivariant.* More generally, if σ and τ are noncharacteristic curves each connecting x_1 to x_2, u along σ and τ are equivariant. Since equivariant functions have the same total increasing and decreasing variations, we conclude that *the total increasing and decreasing variations of a differentiable solution between any pair of characteristics are conserved.*

Denote by $D(t)$ the width of the strip bounded by x_1 and x_2:

$$D(t) = x_2(t) - x_1(t). \tag{4.20}$$

Differentiating (4.20) with respect to t and using (4.19) we obtain

$$\frac{d}{dt}D(t) = \frac{dx_2}{dt} - \frac{dx_1}{dt} = a(u_2) - a(u_1). \tag{4.21}$$

Integrating with respect to t we obtain

$$D(T) = D(0) + [a(u_2) - a(u_1)]T. \tag{4.22}$$

Suppose there is a shock y present in u between the characteristics x_1 and x_2 (see Fig. 10). Since according to (3.13) characteristics on either side of a shock

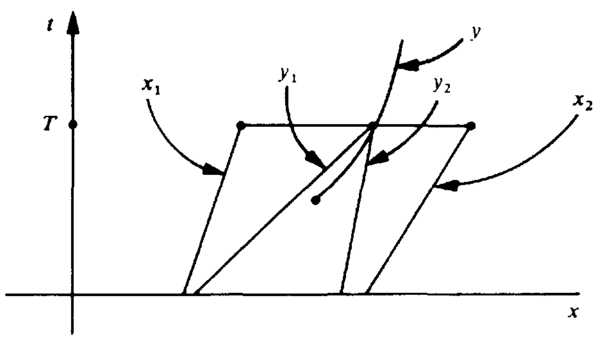

FIG 10

run into the shock, there exist at any given time T, two characteristics y_1 and y_2 which intersect the shock y at exactly time T. Assuming that there are no other shocks present we conclude that the increasing variation of u on $(x_1(t), y_1(t))$, as well as on $(x_2(t), y_2(t))$, is independent of t. According to condition (3.13), $a(u)$ decreases across shocks, so the increasing variation of $a(u)$ along $[x_1(T), x_2(T)]$ equals the sum of the increasing variations of $a(u)$ along $[x_1(0), y_1(0)]$ and along $[y_2(0), x_2(0)]$. This sum is in general less than the increasing variation of u along $[x_1(0), x_2(0)]$; therefore we conclude that *if shocks are present, the total increasing variation of $a(u)$ between two characteristics decreases with time.*

We give now a quantitative estimate of this decrease. Let I_0 be any interval of the x-axis; we subdivide it into subintervals $[y_{j-1}, y_j]$, $j = 1, \cdots, n$, so that $u(x, 0)$ is alternately increasing and decreasing on these intervals (we assume for simplicity that u_0 is piecewise monotonic). We denote by $y_j(t)$ the characteristic issuing from the jth point y_j, with the understanding that if $y_j(t)$ runs into a shock, $y_j(t)$ is continued as that shock.

It is easy to show that for any $t > 0$, $u(x, t)$ is alternately increasing and decreasing on the intervals $(y_{j-1}(t), y_j(t))$. Since across shocks u decreases, the total increasing variation $A^+(T)$ of $a(u)$ across the interval $I(T) = [y_0(T), y_n(T)]$ is

$$\sum_{j\,\mathrm{odd}} a(u_j(T)) - a(u_{j-1}(T)) = A^+(T), \tag{4.23}$$

where $u_{j-1}(T)$ denotes the value of u on the right edge of $y_{j-1}(T)$, $u_j(T)$ denotes the value of u on the left edge of $y_j(T)$; in case that $y_{j-1}(T)$ and $y_j(T)$ are the same, the jth term in (4.23) is zero. Suppose, $y_{j-1}(T) < y_j(T)$; then there exist characteristics $x_{j-1}(t)$ and $x_j(t)$ which start at $t = 0$ inside (y_{j-1}, y_j) and which at $t = T$ run into $y_{j-1}(T)$, and $y_j(T)$ respectively. The value of u along $x_j(t)$ is $u_j(T)$.

Denote $x_j(t) - x_{j-1}(t)$ by $D_j(t)$; according to (4.22)

$$D_j(T) = D_j(0) + [a(u_j) - a(u_{j-1})]T.$$

Summing over j odd and using (4.23) we obtain

$$\sum D_j(T) = \sum D_j(0) + A^+(T)T.$$

Since the intervals $[x_{j-1}(T), x_j(T)] = [y_{j-1}(T), y_j(T)]$ are disjoint and lie in $I(T)$, the sum of their lengths cannot exceed the length $L(T)$ of $I(t)$. So we deduce that

$$A^+(T) \leqq \frac{L(T)}{T}. \tag{4.24}$$

Suppose the initial value of u is periodic, with period p. Then by uniqueness it follows that u is periodic for all p. Take L_0 to be of length p; then $L(T)$ also has length p and we deduce from (4.24) that $A^+(T) \leqq p/T$. Since the increasing variation of a periodic function per period is twice its total variation, we have proved the following theorem.

THEOREM 4.3. *For every space periodic solution u of* (4.11),

$$\frac{\text{total variation of } a(u(T)) \text{ per period}}{\text{period}} \leqq \frac{2}{T}. \tag{4.25}$$

Relation (4.25) shows that the total variation of $a(u)$ per period tends to zero as T tends to ∞. Since the mean value $\bar{u}_0$ of any periodic solution of a conservation law is independent of t, it follows that $u(x, T)$ tends to $\bar{u}_0$ uniformly as T tends to ∞. Suppose $|a'(\bar{u}_0)| = h \neq 0$; it follows from (4.25) that the total variation of $u(T)$ per period is $\leqq 2p/hT$; so we conclude that for T large enough,

$$|u(x, t) - \bar{u}_0| < \frac{p}{ht}. \tag{4.26}$$

Comparison of (4.26) with (4.9) shows that periodic solutions decay *faster* than solutions whose initial values are integrable.

The estimate (4.26) is, in contrast to (4.9), absolute, inasmuch as the right side is independent of the amplitude of the solution.

5. Hyperbolic systems of conservation laws. In this section we shall study systems of conservation laws,

$$\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x}f_i = 0, \qquad i = 1, \cdots, n, \tag{5.1}$$

where each f_i is a function of $u_1, \cdots, u_n$; we shall denote the column vector formed by $u_1, \cdots, u_n$ by u. Carrying out the differentiation in (5.1) we obtain the quasi-linear system

$$u_t + A(u)u_x = 0, \tag{5.2}$$

where the ith row of the matrix A is the gradient of f_i with respect to u. We assume that the system (5.2) is hyperbolic, i.e., that for each value of u the matrix A has n real, distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, labeled in increasing order. Since A depends on u, so do the eigenvalues λ_k and the corresponding right and left eigenvectors r_k and l_k.

Genuine nonlinearity played an important role for single conservation laws; this was the requirement that λ be a nonconstant function of u, i.e., that $\lambda_u \neq 0$. The analogous condition for system is not merely that $\operatorname{grad}_u \lambda_k$ be nonzero, but that it be not orthogonal to r_k, the corresponding eigenvector. If this is so, we call the kth field *genuinely nonlinear*, and normalize r_k so that

$$r_k \cdot \operatorname{grad} \lambda_k = 1. \tag{5.3}$$

If on the other hand $r_k \cdot \operatorname{grad} \lambda_k \equiv 0$, we call the kth characteristic field *linearly degenerate*.

We turn now to the study of piecewise continuous solutions of (5.1) in the integral sense; each of the n conservation laws must satisfy the Rankine–Hugoniot jump condition, i.e.,

$$s[u_k] = [f_k], \qquad k = 1, \cdots, n, \tag{5.4}$$

must hold across every discontinuity, where s is the speed of propagation of the discontinuity.

Next we formulate an *entropy condition* for systems. For single convex (or concave) equations this condition requires that the characteristics on either side of a discontinuity run into the line of discontinuity, which is the case if the characteristic speed on the left is greater, on the right less, than s:

$$\lambda(u_l) > s > \lambda(u_r).$$

For systems we require that *for some index* k, $1 \leqq k \leqq n$,

$$\lambda_k(u_l) > s > \lambda_k(u_r) \tag{5.5'}$$

while

$$\lambda_{k-1}(u_l) < s < \lambda_{k+1}(u_r). \tag{5.5''}$$

These inequalities assert that k characteristics impinge on the line of discontinuity from the left and $n - k + 1$ from the right, a total of $n + 1$. This information carried by these characteristics plus the $n - 1$ relations obtained from (5.4) after eliminating s are sufficient to determine the $2n$ values which u takes on on both sides of the line of discontinuity.

A discontinuity across which (5.4) and (5.5) are satisfied is called a *k-shock.*

We give now a description of all weak k-shocks, i.e., those where u_r and u_l differ little; it is understood that u_l is to the left of u_r.

THEOREM 5.1. *The set of states u_r near u_l which are connected to some given state u_l through a k-shock form a smooth one-parameter family $u_r = u(\varepsilon)$, $-\varepsilon_0 < \varepsilon \leqq 0$, $u(0) = u_l$; the shock speed s also is a smooth function of ε.*

We shall omit the proof of this result but shall calculate the first two derivatives of $u(\varepsilon)$ at $\varepsilon = 0$. Differentiating the jump relation

$$s[u - u_l] = f(u) - f(u_l)$$

we obtain, using the symbol $\dot{} = d/d\varepsilon$,

$$\dot{s}[u] + s\dot{u} = \dot{f} = A\dot{u}. \tag{5.6}$$

At $\varepsilon = 0$ we have $[u] = 0$, so there

$$s(0)\dot{u}(0) = A(u_l)\dot{u}(0),$$

which can be satisfied with $\dot{u} \neq 0$ only if $s(0)$ is an eigenvalue of A:

$$s(0) = \lambda_k(u_l), \tag{5.7}$$

and $\dot{u}(0)$ an eigenvector:

$$\dot{u}(0) = \alpha r_k(u_l). \tag{5.8}$$

By reparametrizing we can make $\alpha = 1$. Differentiating (5.6) once more and setting $\varepsilon = 0$ we obtain (omitting the subscript k)

$$s\ddot{u} + 2\dot{s}\dot{u} = A\ddot{u} + \dot{A}\dot{u}.$$

Substituting (5.7) and (5.8), with $\alpha = 1$ we obtain

$$\lambda\ddot{u} + 2\dot{s}r = A\ddot{u} + \dot{A}r. \tag{5.9}$$

To determine $\dot{s}$ and $\ddot{u}$ we turn to the eigenvalue relation

$$\lambda r = Ar,$$

restricted to $u = u(\varepsilon)$ and differentiate with respect to ε:

$$\lambda\dot{r} + \dot{\lambda}r = A\dot{r} + \dot{A}r.$$

Subtract this from (5.9):

$$\lambda(\ddot{u} - \dot{r}) + (2\dot{s} - \dot{\lambda})r = A(\ddot{u} - \dot{r}). \tag{5.10}$$

Taking the scalar product with the left eigenvector l belonging to λ we obtain

$$2\dot{s} - \dot{\lambda} = 0.$$

Since by (5.3), $\dot{\lambda} = \dot{u}$ grad $\lambda = r$ grad $\lambda = 1$, we obtain

$$2\dot{s} = \dot{\lambda} = 1. \tag{5.11}$$

Equation (5.10) has the solution $\ddot{u} - \dot{r} = \beta r$; by a change of parameter ε we can accomplish that $\beta = 0$; so

$$\ddot{u}(0) = \dot{r}(0). \tag{5.12}$$

It follows from (5.11) that, nodulo terms $O(\varepsilon^2)$,

$$\lambda(\varepsilon) = \lambda(0) + \varepsilon,$$

$$s(\varepsilon) = \lambda(0) + \varepsilon/2.$$

Thus the entropy condition (5.5)

$$\lambda(u_l) = \lambda(0) > s(\varepsilon) > \lambda(\varepsilon) = \lambda(u_r)$$

is satisfied for $\varepsilon < 0$ and not for $\varepsilon > 0$; that is why in Theorem 5.1 the parameter ε is restricted to $\varepsilon \leqq 0$.

Next we turn to an important class of continuous solutions, centered rarefaction waves; these are solutions which depend only on the ratio $(x - x_0)/(t - t_0)$, x_0, t_0 being the center of the wave.

Let u be a rarefaction wave centered at the origin:

$$u(x, t) = h(x/t). \tag{5.13}$$

Let us denote x/t by ξ and differentiation with respect to ξ by $'$; substituting (5.13) into (5.2) we obtain

$$-\frac{x}{t^2}h' + \frac{1}{t}Ah' = 0,$$

which is the same as

$$[A(h) - \xi]h'(\xi) = 0.$$

This is satisfied by

$$\xi = \lambda(h(\xi)), \quad h' = \alpha r(h). \tag{5.14}$$

Using relation (5.3) we obtain, after differentiating the first relation in (5.14) and using the second, that

$$1 = \lambda(h(\xi))' = h' \operatorname{grad} \lambda = \alpha r \operatorname{grad} \lambda = \alpha,$$

where $\lambda = \lambda_k$ is one of the eigenvalues of A; h is called a *k-rarefaction wave.* Setting $\alpha = 1$ in (5.14) we obtain

$$h' = r(h). \tag{5.14'}$$

Abbreviate $\lambda(u_l)$ by λ; the differential equation (5.14') has a unique solution satisfying the initial condition

$$h'(\lambda) = u_l; \tag{5.15}$$

h is defined for all ξ close enough to λ.

Let ε be a number $\geqq 0$, so small that h is defined at $\lambda + \varepsilon$; denote by u_r the value

$$u_r = h(\lambda + \varepsilon). \tag{5.16}$$

We construct now the following piecewise smooth function $u(x, t)$, defined for $t \geqq 0$ (see Fig. 11):

$$u(x,t) = \begin{cases} u_l & \text{for} \quad x < \lambda t, \\ h(x/t) & \text{for} \quad \lambda t \leqq x \leqq (\lambda + \varepsilon)t, \\ u_r & \text{for} \quad (\lambda + \varepsilon)t < x. \end{cases} \tag{5.17}$$

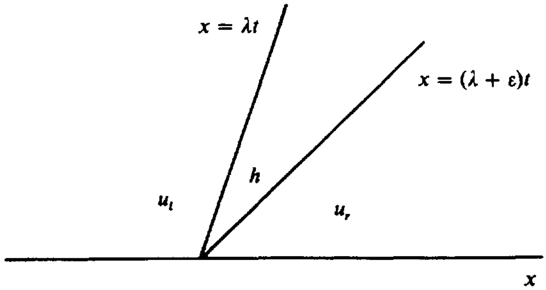

FIG. 11

This function u satisfies the differential equation (5.2) in each of the three regions, and is continuous across the lines separating the regions. We shall say that in u *the states u_l and u_r are connected by a centred k-rarefaction wave.*

THEOREM 5.2. *Given a state u_l, there is a one-parameter family of states $u_r = u(\varepsilon)$, $0 \leqq \varepsilon \leqq \varepsilon_0$, which can be connected to u_l through a centered k-rarefaction wave.*

Theorems 5.1 and 5.2 can be combined.

THEOREM 5.3. *Given a state u_l, it can be connected to a one-parameter family of states $u_r = u(\varepsilon)$, $-\varepsilon_0 < \varepsilon < \varepsilon_0$, on the right of u_l through a centered k-wave, i.e., either a k-shock or a k-rarefaction wave; $u(\varepsilon)$ is twice continuously differentiable with respect to ε.*

The only part that needs proof is the continuity of $du/d\varepsilon$ and $d^2u/d\varepsilon^2$ at $\varepsilon = 0$. From (5.8) and (5.14) we see, since $\alpha = 1$ in both cases, that $du/d\varepsilon = r(u_l)$ for $\varepsilon = \pm 0$; to show that $d^2u/d\varepsilon^2$ is continuous at $\varepsilon = 0$ we differentiate (5.14') with respect to ε to obtain

$$\frac{d^2}{d\varepsilon^2}u(+0) = h''(\lambda) = r'. \tag{5.18}$$

Since $u'(0) = \dot{u}(0)$, we see, comparing (5.18) and (5.12), that $u'' = \ddot{u}$ at $\varepsilon = 0$. This completes the proof of Theorem 5.3.

If the kth characteristic field of (5.1) is degenerate, then (5.1) has discontinuous solutions whose speed of propagation is

$$s = \lambda_k(u_l) = \lambda_k(u_r).$$

These are called contact discontinuities.

We turn now to the so-called *Riemann initial value problem*, where the initial function u_0 is

$$u_0(x) = \begin{cases} u_0 & \text{for} \quad x < 0, \\ u_n & \text{for} \quad 0 < x, \end{cases} \tag{5.19}$$

where u_0 and u_n are two vectors.

THEOREM 5.4. *If the states u_0 and u_n are sufficiently close, the initial value problem* (5.19) *has a solution. This solution consists of $n + 1$ constant states $u_0, u_1, \cdots, u_n$, separated by centered rarefaction or shock waves, one of each family* (see Fig. 12).

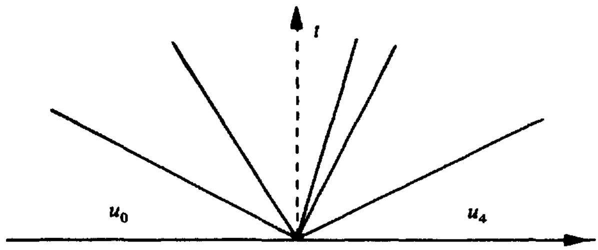

FIG. 12

Proof. The state u_0 can be connected through a 1-wave to a one-parameter family of states $u_1(\varepsilon_1)$ to the right of u_0; u_1 in turn can be connected through a 2-wave to a one-parameter family $u_2(\varepsilon_1, \varepsilon_2)$ of states to the right of u_1, etc. Thus u_0 can be connected through a succession of n waves to an n-parameter family of states $u_n(\varepsilon_1, \cdots, \varepsilon_n)$. By (5.8)–(5.14),

$$\frac{\partial u}{\partial \varepsilon_j} = r_j;$$

since the r_j are linearly independent, it follows from the implicit function theorem that a small ε-ball is mapped one-to-one onto a neighborhood of u_0; this completes the proof of Theorem 5.4.

We describe now a method developed by James Glimm for solving any initial value problem $u(x, 0) = u_0(x)$ where the oscillation of u_0 is small. The solution u is obtained as the limit as $h \to 0$ of approximate solutions u_h constructed as follows:

(I) $u_h(x, 0)$ is a *piecewise constant approximation* to $u_0(x)$:

$$u_h(x, 0) = m_j \quad \text{for} \quad jh < x < (j + 1)h, \qquad j = 0, \pm 1, \cdots, \tag{5.20}$$

where m_j is some kind of mean value of $u_0(x)$ over the interval $(jh, (j + 1)h)$.

(II) For $0 \leqq t < h/\lambda$, $u_h(x, t)$ is the exact solution of (5.1) with initial values $u_h(x, 0)$ given by (5.20); here λ is an upper bound for $2|\lambda_k(u)|$. This exact solution is constructed by solving the Riemann initial value problems

$$u(x,0) = \begin{cases} m_{j-1} & \text{for } \ x < jh, \\ m_j & \text{for } \ jh < x, \end{cases} \tag{5.21$_j$}$$

$j = 0, \pm 1, \cdots$. Since the oscillation of u_0 is small, m_{j-1} and m_j are close and so according to Theorem 5.4 this initial value problem has a solution consisting of constant states separated by shocks or centered rarefaction waves issuing from the points $x = jh$, $t = 0$ (see Fig. 13). As long as

$$t < \frac{h}{\lambda} \tag{5.22}$$

these waves do not intersect each other and so the solutions of the initial value problem (5.21$_j$) can be combined into a single exact solution u_h.

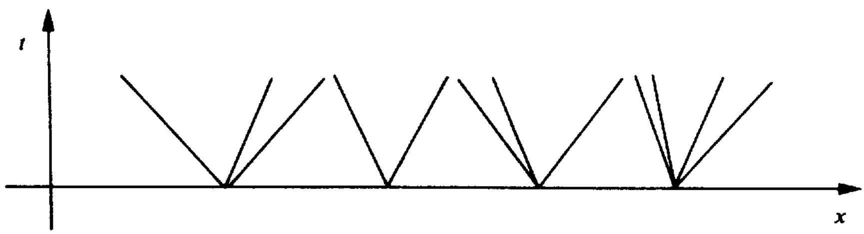

FIG. 13

(III) We repeat the process, with $t = h/\lambda$ as new initial time in place of $t = 0$.

It is not at all obvious that this process yields an approximate solution u_h which is defined for all t; to prove this one must show that the oscillation of $u(x, nh)$ remains small, uniformly for $n = 1, 2, \cdots$ so that the Riemann problems (5.21$_j$) can be solved, and so that λ does not tend to ∞. This estimate turns out to depend very sensitively on the kind of averaging used to compute the mean values m_j. In the scheme introduced by Glimm the quantities m_j are computed as follows: A sequence of random numbers $\alpha_1, \alpha_2, \cdots$, uniformly distributed in $[0, 1]$, is chosen; m_j^n, the mean value of $u(x, nh/\lambda)$ over the interval $(jh, (y + 1)h)$ is taken to be

$$m_j^n = u(jh + \alpha_n h, nh/\lambda). \tag{5.23}$$

Glimm shows:

(A) Given any ε, we can choose η so small that if the oscillation and total variation of u_0 are $< \eta$ then for any t, the oscillation and total variation of $u(x, t)$ along any space-like line is $< \varepsilon$.

(B) A subsequence of u_h converges in the L_1 norm with respect to x, uniformly in t, to a limit u.

(C) For almost all choices of the random sequence $\{\alpha_n\}$, this limit u is a solution in the integral sense of the conservation law (5.1).

For proof we refer to Glimm's paper; here we merely point out how Glimm's scheme treats a particularly simple initial value problem, a Riemann initial value problem:

$$u_0(x) = \begin{cases} u_l & \text{for} \quad x < 0, \\ u_r & \text{for} \quad 0 < x, \end{cases}$$

where u_l and u_r are so chosen that the exact solution u consists of the two states u_l, u_r separated by a single shock

$$u(x, t) = \begin{cases} u_l & \text{for} \quad x < st, \\ u_r & \text{for} \quad st < x, \end{cases} \tag{5.24}$$

where s is the speed with which the shock separating the two states propagates. By assumption, $\lambda > |s|$. Let us assume that $0 < s$; then Glimm's recipe (5.23) gives

$$u_h(x, h/\lambda) = \begin{cases} u_l & \text{for} \quad x < J_1 h, \\ u_r & \text{for} \quad J_1 h < x, \end{cases}$$

where

$$J_1 = \begin{cases} 1 & \text{if} \quad \alpha_1 < s/\lambda, \\ 0 & \text{if} \quad s/\lambda < \alpha_1. \end{cases}$$

Repeating this analysis n times we obtain

$$u_h(x, nh/\lambda) = \begin{cases} u_l & \text{for} \quad x < J_n h, \\ u_r & \text{for} \quad J_n h < x, \end{cases} \tag{5.25}$$

where J_n = number of α_j, $j = 1, \cdots, n$, which satisfy

$$\alpha_j < \frac{s}{\lambda}. \tag{5.26}$$

Since $\{\alpha_j\}$ is a uniformly distributed random sequence,

$$\frac{J_n}{n} \to \frac{s}{\lambda} \tag{5.27}$$

with probability 1; this shows that the approximate solution given by (5.25) tends almost certainly to the exact solution given by (5.24).

Note that if, instead of using random sequences we use a *single sequence* of *equidistributed* numbers $\{\alpha_j\}$, i.e., numbers for which (5.27) holds, we conclude that u_h tends to u as $h \to 0$.

We conclude by stating precisely the existence theorem whose proof was outlined above, and by stating some open problems.

THEOREM 5.5. *The initial value problem for the system of conservation laws* (5.1) *has a solution for all t provided that the initial function u_0 has sufficiently small oscillation and total variation.*

What is lacking at present is a proof that the solution u constructed by Glimm's scheme satisfies the entropy condition, and that it is uniquely determined by u_0. Some remarks about both points will be made in § 6.

Another outstanding problem is to remove the requirement that u_0 have small oscillation.

Inequalities (3.14), (3.38) and (5.5) are criteria which reject certain discontinuities as physically unrealizable even though the conservation laws are satisfied across them; we designated these criteria as *entropy conditions*. We shall introduce now a notion of *entropy* which can be related to these criteria.

We start with a system of conservation laws (5.1). Let U be some function of $u_1, \cdots, u_n$. When does U satisfy a conservation law, i.e., a law of the form

$$U_t + F_x = 0, \tag{5.28}$$

where F is some function of $u_1, \cdots, u_n$? Carrying out the differentiation in (5.28) we obtain

$$\operatorname{grad} U \cdot u_t + \operatorname{grad} F \cdot u_x = 0,$$

where the gradient is with respect to u. To deduce this from (5.2),

$$u_t + Au_x = 0, \tag{5.2}$$

we multiply (5.2) on the left with grad U; (5.28) results if and only if the relation

$$\operatorname{grad} U \cdot A = \operatorname{grad} F \tag{5.29}$$

holds. This is a system of n partial differential equations for U and F; for $n \geqq 2$ it is overdetermined and has no solution in general; there are, however, special cases of some importance with a nontrivial solution, for example, in gas dynamics. A general class of equations where a solution exists are the symmetric ones, i.e., when A is a symmetric matrix. In this case,

$$\frac{\partial f_j}{\partial u_l} = \frac{\partial f_l}{\partial u_j}. \tag{5.30}$$

Relation (5.30) is the compatability relation for the existence of a function $g(u)$ satisfying

$$\frac{\partial g}{\partial u_l} = f_l.$$

It is then easy to verify that

$$U = \sum u_j^2 \quad \text{and} \quad F = \sum u_j f_j - g \tag{5.31}$$

satisfy (5.29).

The role of entropy conditions is to distinguish those discontinuous solutions which are physically realizable from those which are not. Another way to characterize the physically realizable solutions is to identify them as limits of solutions of equations in which a small dissipative mechanism has been added to the laws already embodied in (5.1). A particular example of such a dissipative mechanism is *artificial viscosity*; here the augmented equation is

$$u_t + Au_x = \lambda u_{xx}, \qquad \lambda > 0. \tag{5.32}$$

Multiply this by grad U; if (5.29) is satisfied we obtain

$$U_t + F_x = \lambda \operatorname{grad} U \cdot u_{xx}. \tag{5.33}$$

We have the identity

$$U_{xx} = \operatorname{grad} U \cdot u_{xx} + U_{ij} u_x^i u_x^j,$$

where

$$U_{ij} = \frac{\partial^2 u}{\partial u^i \partial u^j}.$$

Suppose U is *convex*, i.e., the matrix of its second derivatives is positive definite; then we deduce that

$$U_{xx} \geqq \operatorname{grad} U \cdot u_{xx};$$

substituting this into (5.33) we obtain, since $\lambda > 0$, that

$$U_t + F_x \leqq \lambda U_{xx}.$$

Let $\lambda \to 0$; the right side tends to zero in the sense of distributions and we deduce the following theorem.

THEOREM 5.6. *Let* (5.1) *be a system of conservation laws which implies an additional conservation law* (5.28); *suppose that U is strictly convex. Let $u(x, t)$ be a distribution solution of* (5.1) *which is the limit, boundedly, a.e., of solutions of* (5.32) *containing the artificial viscous term. Then u satisfies the inequality*

$$U(u)_t + F(u)_x \leqq 0. \tag{5.34}$$

The following are immediate consequences of (5.34):

(a)

$$\int U(t)\, dx, \tag{5.35}$$

if finite, is a decreasing function of t.

(b) *Suppose u is piecewise continuous; then across a discontinuity*

$$s[U_l - U_r] - [F_l - F_r] \leqq 0. \tag{5.36}$$

We shall call conditions (5.34) and (5.36) *entropy conditions*; to justify the name we have to show compatability with the previous designations.

THEOREM 5.7. *Suppose that the system of conservation laws* (5.1) *is hyperbolic and genuinely nonlinear in the sense of* (5.3). *Suppose there is a strictly convex function U of u which satisfies the additional conservation law* (5.28). *Let u be a solution of* (5.1) *in the integral sense which has a discontinuity propagating with* *speed s. Suppose that the values on the two sides of the discontinuity are close; then the entropy condition* (5.5) *holds if and only if the entropy condition* (5.36) *holds.*

Proof. According to Theorem 5.1, the states u_r which can be connected to a given state u_l through a k-shock, form a one-parameter family of states $u_r = u(\varepsilon)$, $-\varepsilon_0 < \varepsilon < 0$. Denote by $E(\varepsilon)$ the quantity on the left of (5.36):

$$E(\varepsilon) = s[U(u_r) - U(u_l)] - F(u_r) + F(u_l).$$

A brief calculation using (5.8) and (5.11), and which we omit, shows that the values of the first two derivatives with respect to ε of $E(\varepsilon)$ are zero at $\varepsilon = 0$, and the value of the third derivative at $\varepsilon = 0$ is

$$\dddot{E} = \tfrac{1}{2}{}'rU''r.$$

Here ${}'r$ is the transpose of the right eigenvector r and U'' is the matrix of second derivatives of U. Since U is convex, $\dddot{E}$ is positive; this shows that $E(\varepsilon)$ is an increasing function of ε near $\varepsilon = 0$. But then $E(\varepsilon)$ is negative for ε negative; since by Theorem 5.1 the shock condition (5.5) restricts $u_n(\varepsilon)$ to negative values of ε, Theorem 5.7 follows.

Next we show that the entropy condition (5.34) is equivalent to the entropy condition in the large (3.38) imposed on solutions of single equations. We note that in this case U can be taken to be any function of u; F can be determined from (5.29) by integration.

THEOREM 5.8. *The entropy condition* (3.38) *is satisfied if and only if* (5.36) *is satisfied for all U, F which satisfy* (5.29) *and where U is convex.*

Proof. Suppose $u_l < u_r$; let z be any number between u_l and u_n, and set

$$U(u) = \begin{cases} 0 & \text{for} \quad u < z, \\ u - z & \text{for} \quad z \leqq u. \end{cases}$$

It follows from (5.29) that

$$F(u) = \begin{cases} 0 & \text{for} \quad u < z, \\ f(u) - f(z) & \text{for} \quad z \leqq u. \end{cases}$$

Substituting these into (5.36) and using the jump relation

$$s = \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

we obtain, after rearrangement,

$$f(z) \geqq \frac{u_n - z}{u_r - u_l} f(u_l) + \frac{z - u_l}{u_r - u_l} f(u_r),$$

which is precisely condition (3.38_{ii}); the other case can be handled similarly.

6. Pairs of conservation laws. More is known about systems of two conservation laws than systems consisting of more than two. What is special about systems of only two equations is the existence of Riemann invariants, which we now proceed to describe.

We write the system as

$$u_t + f_x = 0, \tag{6.1}$$
$$v_t + g_x = 0,$$

where f and g are functions of u and v. Carrying out the differentiation in (6.1) we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + A \begin{pmatrix} u \\ v \end{pmatrix}_x = 0, \tag{6.2}$$

where

$$A = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}.$$

We assume that (6.2) is hyperbolic, which means that the matrix A has real and distinct eigenvalues; we denote them by λ and ρ, so arranged that $\lambda < \rho$; of course λ and ρ both are functions of u and v. We denote the corresponding left and right eigenvector by l and r; that is

$$l_\lambda A = \lambda l_\lambda, \qquad A r_\lambda = \lambda r_\lambda$$

and

$$l_\rho A = \rho l_\rho, \qquad A r_\rho = \rho r_\rho.$$

The eigenvectors, too, depend on u and v. We shall consider functions w of u and v which satisfy the first order equation

$$\operatorname{grad} w \cdot r_\lambda = 0. \tag{6.3$_\lambda$}$$

This equation asserts that w is constant along the trajectories of the vector field r_λ. We can construct solutions of (6.3) by taking a curve C which is not tangent to r_λ at any point, and assigning arbitrary values for w along it. We shall choose w to be *strictly increasing* along C. The value of u is then determined along every trajectory intersecting C, and w has distinct values along distinct trajectories.

The function z of u, v is defined analogously as solution of

$$\operatorname{grad} z \cdot r_\rho = 0. \tag{6.3$_\rho$}$$

Since w has distinct values along distinct r_λ-trajectories, and z has distinct values along r_ρ-trajectories, and since in a simply connected domain of u, v space an r_ρ-trajectory intersects an r_λ-trajectory in at most one point, it follows that the mapping

$$u, v \to w, z$$

is one-to-one over any simply connected domain.

It is well known that the left and right eigenvectors of a matrix with distinct eigenvalues are biorthogonal. It is here that we exploit that $n = 2$: since by (6.3) grad w is orthogonal to r_λ, it follows that grad w is a left eigenvector with eigenvalue ρ:

$$\text{grad}\, wA = \rho\, \text{grad}\, w. \tag{6.4$_\rho$}$$

Similarly,

$$\text{grad}\, zA = \lambda\, \text{grad}\, z. \tag{6.4$_\lambda$}$$

Multiply (6.2) by grad w; using the above relation, and the chain rule, we obtain

$$w' = w_t + \rho w_x = 0, \tag{6.5}$$

where $'$ denotes differentiation in the direction

$$\frac{dx}{dt} = \rho. \tag{6.6}$$

Similarly,

$$\dot{z} = z_t + \lambda z_x = 0, \tag{6.7}$$

where $\cdot$ is differentiation in the direction

$$\frac{dx}{dt} = \lambda. \tag{6.8}$$

Curves satisfying (6.6) and (6.8) are called λ and ρ *characteristics* respectively. Relations (6.5) and (6.7) can be stated in words as follows.

As functions of x and t, *w is constant along ρ-characteristics, z is constant along λ-characteristics*. After their discoverer, w and z are called *Riemann invariants.*

In § 3 about single conservation laws we gave a geometric argument for the nonexistence of continuous solutions beyond a certain time. One ingredient of that proof was the constancy of u along characteristics; the other was the fact that characteristics are straight lines. The first ingredient is present here: w and z are constant along characteristics, but it is no longer true that characteristics are straight lines; so the simple geometric reasoning given in § 3 cannot be extended to systems. We present now a different nonexistence proof for a single equation which is capable of generalization.

The equation (3.3) satisfied by u is

$$u_t + a(u)u_x = 0;$$

differentiate this with respect to x:

$$u_{tx} + au_{xx} + a_u u_x u_x = 0.$$

Abbreviate u_x by q; the above can be written then as

$$q' + a_u q^2 = 0, \tag{6.9}$$

where q' abbreviates the directional derivative

$$q' = q_t + aq_x.$$

Equation (6.9) is an ordinary differential equation for q along the characteristic and can be integrated explicitly:

(6.10) $$q(t) = \frac{q_0}{1 + q_0 kt},$$

where $q_0 = q(0)$ and $k = a'(u)$ constant along the characteristic. This formula shows that if $q_0 k > 0$, $g(t)$ is bounded for all $t > 0$, while if $q_0 k < 0$, $q(t)$ blows up at $t = -1/q_0 k$.

We imitate the above proof for the system (6.5), (6.7) as follows. Differentiate (6.5) with respect to x:

$$w_{tx} + \rho w_{xx} + \rho_w w_x^2 + \rho_z w_x z_x = 0.$$

Abbreviating w_x by p we can put this as

(6.11) $$p' + \rho_w p^2 + \rho_z p z_x = 0.$$

From (6.7) we deduce that

(6.12) $$z_x = \frac{z'}{\rho - \lambda};$$

substituting this into (6.11) we obtain

(6.13) $$p' + \rho_w p^2 + \frac{\rho_z}{\rho - \lambda} z' p = 0.$$

Let h denote a function of w and z which satisfies

$$h_z = \frac{\rho_z}{\rho - \lambda}.$$

Since according to (6.5), $w' = 0$,

$$h' = h_w w' + h_z z' = \frac{\rho_z}{\rho - \lambda} z'.$$

Substituting this into (6.13) we obtain

$$p' + \rho_w p^2 + h' p = 0.$$

Multiply this by e^h; using the abbreviations

(6.14) $$e^h p = q, \qquad e^{-h} \rho_w = k,$$

the resulting equation can be rewritten as

(6.15) $$q' + kq^2 = 0.$$

This is an ordinary differential equation for q along each ρ-characteristic, similar to (6.9) except that the coefficient k of q^2 is not constant. Nevertheless an explicit formula for q can be written:

(6.16) $$q(t) = \frac{q_0}{1 + q_0 K(t)},$$

where $q_0 = q(0)$ and

$$K(t) = \int_0^t k\,dt,$$

the integration along the ρ-characteristic.

Clearly, the boundedness or not of $q(t)$ hinges on whether $q_0 K(t)$ ever takes on the value -1. This is easily analyzed if $\rho_w \neq 0$ anywhere; since the sign of w is arbitrary, we might as well assume that

$$\rho_w > 0. \tag{6.17}$$

Suppose the initial values of w, z are bounded: $|w|, |z| < M$. The same inequalities hold for all $t > 0$, since the value of w or z at any point equals the value of w or z at that point on the initial line to which P can be connected by a ρ, or λ characteristic. Once we know that (w, z) stays for all time in a bounded set, we can conclude from (6.17) that the function k defined as $e^{-h}\rho_w$ in (6.14) is bounded from below for all t and x by a positive constant k_0. Its integral K then satisfies

$$K(t) \geqq k_0 t \quad \text{for all} \quad t \geqq 0.$$

Substituting this into (6.16) we conclude that if $q_0 > 0$, $q(t)$ stays bounded, if $q_0 < 0$, $q(t)$ becomes unbounded after a finite time. We see from (6.14) that the sign of q_0 is the same as that of p_0, the initial value of w_x. So we can summarize what we have proved as follows.

THEOREM 6.1. *Suppose condition* (6.17) *is satisfied for a system of equations* (6.2). *Let u, v be a solution whose initial values are bounded; then if $w_x(x, 0)$ is <0 at any point, the derivatives of the solution become unbounded after a finite time.*

A similar result holds with respect to the other variable z, and the following *converse holds*:

Suppose that for a system (6.2), $\rho_w > 0$ *and* $\lambda_z > 0$. *Suppose u, v is a solution of* (6.2) *whose initial values are bounded, and suppose that both $w(x, 0)$ and $z(x, 0)$ are increasing functions of x. Then the first derivatives of the solution remain uniformly bounded, and the solution exists and is differentiable for all $t > 0$.*

Remark. It is easy to verify that the condition $\rho_w \neq 0$ is the same as the genuine nonlinearity condition (5.3).

We turn now to solutions with shocks. Since there are two families of characteristics, there are two families of discontinuities; we shall refer to them as ρ-shocks and λ-shocks.

How does the Riemann invariant w change across a λ-shock? According to Theorem 5.1, the states u_r which can be connected to u_l across a λ-shock form on one-parameter family $u(\varepsilon)$, $\varepsilon < 0$; the first derivative of u with respect to ε is given by (5.8):

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}(0) = r_\lambda.$$

Let us calculate $\dot{w}$:

$$\dot{w} = \text{grad}\, w \cdot \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \text{const. grad}\, w \cdot r_\lambda,$$

which according to (6.4_λ) is zero.

A similar calculation, based on (5.12), shows that also

$$\ddot{w} = 0.$$

However $\dddot{w}$ is in general not zero; if we impose the requirement that $\dddot{w} \neq 0$, say

$$\dddot{w} > 0, \tag{6.18}$$

we conclude that, at least for weak shocks,

$$w(u_l) > w(u_r). \tag{6.19}$$

Consider a solution containing a finite number of weak shocks. Let (x, t) be any point, $t > 0$; draw a backward ρ-characteristic C through this point. According to the shock condition (5.5), C cannot run into a ρ-shock; so C can be continued all the way down to the initial line. C will intersect a finite number of λ-shocks; between two points of intersection w is constant. Since $\lambda < \rho$, it follows from (6.19) that w increases along C as t decreases. So we conclude:

If (6.18) *holds, then*

$$w(x, t) < w(y, 0), \tag{6.20}$$

where y is the point where the ρ-characteristic through (x, t) intersects the line $t = 0$.

We turn now to the asymptotic behavior of solutions for large t. This problem was studied for single conservation laws in § 4; the main tool there was the conservation of increasing and decreasing variation of continuous solutions between two characteristics. Thanks to the existence of Riemann invariants, we have a conservation of variation of w between ρ-characteristics and of z between λ-characteristics, valid for continuous solutions. We saw that for solutions of a single conservation law with shocks the variation between two characteristics decreases as t increases; the same argument applied to the Riemann invariants shows that the presence of ρ-shocks causes the variation of w between characteristics to decrease with increasing time, and similarly the presence of λ-shocks causes the variation of z to z to decrease. We have however the additional task of assessing the effect of λ-shocks on the variation of w and of ρ-shocks on z. This has been carried out in Glimm–Lax for solutions whose oscillation is small. The precise result proved there is the following.

THEOREM 6.2. *Suppose condition* (6.18), *and an analogous condition for z, is satisfied for a system of conservation laws* (6.1). *Then the initial value problem for* (6.1) *has a solution for all bounded, measurable initial data whose oscillation is small enough. The total variation of this solution on an interval of length t at time t is bounded by a constant. For periodic solutions, the total variation of u and v per period decays as*

$$\frac{\text{const}}{t}.$$

Not a great deal is known about uniqueness; Oleinik has studied solutions of systems of the special form

$$u_t - v_x = 0, \qquad v_t + f(u)_x = 0,$$

where f is an increasing function of u. She has shown that solutions which contain a finite number of shocks and centered verifaction waves are uniquely determined by their initial data.

We turn now to the entropy condition (5.36) decribed in the last section:

$$U_t + F_x \leqq 0$$

for all $U(u, r)$ and $F(u, r)$ which satisfy (5.29):

$$\text{grad}\, U \cdot A = \text{grad}\, F. \tag{6.21}$$

We can eliminate F from this system of equations by differentiation; we get a homogeneous second order equation for U:

$$aU_{uu} + bU_{uv} + cU_{vv} = 0, \tag{6.22}$$

where

$$a = -f_v, \qquad b = f_u - g_v, \qquad c = g_u. \tag{6.23}$$

It is easy to verify that if the original nonlinear system (6.1) is hyperbolic, so is the linear system (6.21), and the derived equation (6.22). The question is: does a second order hyperbolic equation (6.22) have convex solutions? It is fairly easy to show that in the small the answer is "yes," on the basis of this observation: If the real symmetric matrix a_{ij} is indefinite, there exists a positive definite matrix U_{ij} such that

$$\sum a_{ij} U_{ij} = 0.$$

We show now that the compatibility equation (6.21) has solutions where U is convex, provided that a certain condition, see (6.31) below, is satisfied; we do not claim that this condition is necessary.

We shall construct families of solutions depending on a parameter k in this fashion:

$$U = e^{k\varphi} V + U_N, \qquad F = e^{k\varphi} H + F_N, \tag{6.24}$$

where

$$V = \sum_0^N V_j/k^j, \qquad H = \sum_0^N H_j/k^j. \tag{6.25}$$

φ, V_j and H_j are functions of u and v. U_N and F_N are functions of u, v and k as well, of order $k^{-N} e^{k\varphi}$, i.e.,

$$|D^j(F_N + U_N)| \leqq O(e^{k\varphi}/k^{N-j}). \tag{6.26}$$

Expansions of this sort, with $i\varphi$ in place of φ, are customary in geometrical optics.

The formal construction in the real case is the same as in the imaginary case: We substitute (6.24), (6.25) into (6.21), divide by $e^{k\varphi}$ and equate like powers of k to obtain

$$V_0(\operatorname{grad} \varphi)A = H_0 \operatorname{grad} \varphi, \tag{6.27}$$

$$V_j(\operatorname{grad} \varphi)A + V_{j-1} = H_j \operatorname{grad} \varphi + H_{j-1}, \tag{6.28}$$

$j = 1, 2, \cdots, N$. We solve these recursively. Equation (6.27) asserts that $\operatorname{grad} \varphi$ is a left eigenvector of A, with eigenvalue H_0/V_0. Such a function is called a *phase function* in geometrical optics; according to (6.4) this condition means that φ is a *Riemann invariant.*

Having found φ, we substitute it into (6.28) which we solve recursively, assigning arbitrary initial values for U_j on a noncharacteristic curve. If the initial value for U_0 is chosen to be positive, U_0 will be positive everywhere.

We have to solve a first order system to determine U_N and F_N; the inhomogeneous term in this system is $O(e^{k\varphi}/k^N)$ and, with the proper choice for an initial curve, U_N and V_N will satisfy (6.26).

When is the function U defined by (6.24), (6.25) convex for k large enough, i.e., when is $Q = \xi^2 U_{un} + \xi\eta U_{uv} + \eta^2 U_{vv}$ positive for all ξ, η? The answer can be read off from the first two leading terms of Q in its asymptotic expansion in powers of k. The coefficient of $k^2 e^{k\varphi}$ is

$$(\varphi_u^2 \xi^2 + 2\varphi_u \varphi_v \xi\eta + \varphi_v^2 \eta^2)U_0,$$

which is equal to

$$(\varphi_u \xi + \varphi_v \eta)^2 U_0. \tag{6.29}$$

As we have remarked before, U_0 can be chosen to be positive throughout; therefore, the above form is positive except along

$$(\xi, \eta) = (-\varphi_v, \varphi_u). \tag{6.30}$$

The coefficient of $ke^{k\varphi}$ consists of three terms; two of them are zero along (6.30); the remaining one is

$$\varphi_{uu}\varphi_v^2 - 2\varphi_{uv}\varphi_u\varphi_v + \varphi_{vv}\varphi_u^2. \tag{6.31}$$

We make the assumption that there exists a Riemann invariant φ for which (6.31) is positive; if this is so, U given by (6.24) is convex for k large enough.

If (6.31) is positive, the function

$$\psi(u, v) = e^{k\varphi}$$

is convex; since ψ is a function of the Riemann invariant φ it is itself another Riemann invariant, and so our result can be formulated thus: *If there exists a convex Riemann invariant ψ in a domain of the (u, v)-plane, there exist functions of the form*

$$\begin{aligned} U &= e^{k\psi}\{U_0 + O(1/K)\}, \\ F &= e^{k\psi}\{F_0 + O(1/k)\}, \end{aligned} \tag{6.32}$$

which satisfy (6.21); *furthermore U is convex for k large enough.*

What can we deduce from the entropy condition

$$U_t + F_x \leqq 0 \tag{6.33}$$

for functions of the form (6.32)?

THEOREM 6.3. *Let u, v be a solution in the integral sense of the conservation laws* (6.1), *which satisfies the entropy condition* (6.33) *for all U, F of the form* (6.32), *k large enough. Then*

$$\max_x \psi(u(x, t), v(x, t))$$

is a decreasing function of t.

Sketch of proof. Integrate (6.33) over a lens-shaped region contained between S_1 and S_2 (see Fig. 14). We obtain

$$\int_{S_2} (Un_t + Fn_x)\, ds \leqq \int_{S_1} U\, dx. \tag{6.34}$$

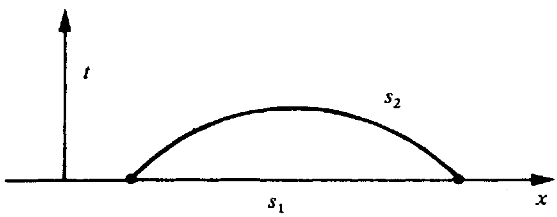

FIG. 14

If S_2 is so chosen that

$$n_t U^0 + n_x F^0 > 0,$$

the kth root of the left side of (6.34) tends as $k \to \infty$ to the maximum of ψ on S_2, while the kth root of the right side tends to the maximum of ψ on S_1. The resulting inequality proves Theorem 6.3, and even a little more.

Remark. The conclusion of Theorem 6.3 agrees with the statement (6.20), deduced under the assumption (6.18). It turns out that inequality (6.18) is equivalent with the positivity of (6.31).

Notes.

1. *Quasilinear equations.* The energy inequality for symmetric hyperbolic systems is due to Friedrichs and Lewy, for nonsymmetric hyperbolic systems, see Leray and Gårding, and Calderon. The existence theorem using the contractive character of $\mathcal{T}$ is due to Schauder. For a more detailed discussion of these approaches, see Chapter VI of Courant and Hilbert.

For the case of functions of one space variable one can employ estimates in the maximum norm instead of the energy norm. This is done as follows: differentiate

(1.1) with respect to x; denoting u_x by p we obtain a relation of the following sort:

$$p_t + Ap_x + Q(p) = 0, \tag{1}$$

where Q is a quadratic function of p. Denote by l_j the left eigenvector of A:

$$l_jA = \lambda_j l_j, \qquad j = 1, \cdots, n. \tag{2}$$

Multiplying (1) by l_j we obtain the following relation for $p_j = l_j p$:

$$\partial_t p_j + \lambda_j \partial_j p_j + q_j(p) = 0, \tag{3}$$

where q_j is some quadratic function of the vector p. The first two terms in (3) are a directional derivative of p_j in a so-called characteristic direction, given by

$$\frac{dx}{dt} = \lambda_j.$$

Integrating (3) along the jth characteristic connecting the point (x, t) to some point on the initial line $t = 0$ we obtain

$$p_j(x, t) = p_j(0) + \int_0^t q_j(p)\, dt. \tag{4}$$

Denote by $M(T)$ the maximum of all $p_j(x, t)$, $j = 1, \cdots, n$, $-\infty < x < \infty$, $0 \leqq t \leqq T$; we obtain from (4) that

$$M \leqq M(0) + cTM^2. \tag{5}$$

From this we deduce easily that

$$M(T) \leqq \frac{1}{2cT}\{1 - \sqrt{1 - 4cTM(0)}\}, \tag{6}$$

valid as long as

$$T \leqq \frac{1}{4cM(0)}. \tag{7}$$

Using this estimate for the first derivatives of solutions one can show that the initial value problem (1.1), (1.3) has a solution in the time interval (7), where $M(0)$ is defined as

$$\max |l_j \partial_j u_0(x)|.$$

The nonexistence Theorem 6.1 shows that the restriction (7) is, roughly, necessary unless one puts conditions on the initial values as in Theorem 6.2.

2. *Conservation laws.* An important class of hyperbolic systems of conservation laws are the ones governing the flow of compressible, nonviscous non-heat conductive fluids. There are five conserved quantities: mass, momenta and energy per unit volume:

ρ = mass per unit volume = density,
M = momenta per unit volume = ρV, where $(u, v, w) = V$ is flow velocity,
E = energy per unit volume = internal + kinetic energy = $\rho e + \frac{1}{2}\rho V^2$, where e = interval energy per unit mass and $V^2 = u^2 + v^2 + w^2$.

The fluxes are partly due to material being transported across the boundary with the velocity of the flow; for the momenta there is an additional flux due to the momentum imparted by the pressure force at the boundary, and there is an energy flux due to the work done by the pressure force at the boundary. If there is no heat conduction this accounts for all energy changes. For a nonviscous fluid the pressure is a scalar p, exerted equally in all directions. The formulas for the fluxes are:

$$\text{mass flux} = \rho V = M,$$

$$\text{momentum flux} = \begin{cases} \rho u V + (p, 0, 0), \\ \rho v V + (0, p, 0), \\ \rho w V + (0, 0, p), \end{cases}$$

$$\text{energy flux} = (E + p)V.$$

Internal energy e, pressure p and density ρ are related by an equation of state:

$$p = p(e, \rho).$$

Using this the fluxes can be expressed in terms of ρ, M and E.

The jump relations for shocks in gas dynamics were first stated by Riemann, incorrectly, for he conserved entropy instead of energy. The correct relations were found by Rankine and by Hugoniot.

Other important nonlinear hyperbolic systems of conservation laws are the equations governing the motion of a shallow layer of water, and the equations of hydrodynamic flows (see Courant and Hilbert, Chapter VI).

3. *Single conservation laws.* The starting point of the rigorous theory of single conservation laws has been a paper by Hopf in 1950, where the explicit formula stated in Theorem 3.1 was given in the special case of the quadratic conservation law $f(u) = \frac{1}{2}u^2$. The formula for arbitrary convex f is stated in Lax (1954), and analyzed in Lax (1957).

The revealing Theorems 3.4 and 3.5 about the decrease of the L_1 norm of the difference of two solutions are due to Barbara Quinn; L_1 contraction also plays a role in the work of Oleinik (1957). Condition (3.38) is due to Oleinik (1959); she showed that solutions satisfying (3.38) are uniquely determined by their initial data, and Kalashnikov proved that solutions of (3.39) converge as $\lambda \to 0$ to a solution which satisfies (3.38).

There is a parallel theory of single conservation laws in n space variables. Existence theorems are contained in Conway and Smoller, Volpert, Krushkov (1969), and Kotlow. A uniqueness theorem for piecewise continuous solutions has been given by Douglis and by Quinn; a more general uniqueness theorem has been given by Krushkov.

4. *The decay of solutions.* In his 1950 paper Hopf studied the large time behavior of solutions of quadratic conservation laws; the extension to any convex p is given in Lax (1957). The more refined Theorem 4.1, and Theorem 4.2 about the two and

only conserved quantities is given in Lax (1970). The law of decrease of increasing variation, and the method for proving it is taken from Glimm and Lax (see also Lax (1972)). For the nonconvex case, see Dafermos (1972).

5. *Hyperbolic systems of conservation laws.* The shock condition (5.5) and Theorems 5.1, 5.2, 5.3, 5.4 are given in Lax (1957). In case the kth field is linearly degenerate, i.e., $\operatorname{grad} \lambda_k \cdot r_k \equiv 0$, there exist discontinuous solutions where the discontinuity is a k-characteristic with respect to either side. Such discontinuities are called *contact discontinuities*. It can be shown that solutions with contact discontinuities only are the limits of continuous solutions.

Foy has shown that if u_l and u_r can be connected by a weak shock, then they can be connected by a viscous profile, i.e., a plane wave solution of the equation

$$u_t + f_x = \lambda u_{xx}$$

with artificial viscous term. By *plane wave* we mean a solution of the form

$$u(x, t) = \frac{v(x - st)}{\lambda},$$

where v is independent of λ and satisfies the ordinary differential equation

$$-sv' + f(v)' = v''.$$

Conley and Smoller have studied viscous profiles for strong shocks.

The main tool in Glimm's existence theorem, beside the difference scheme, is a functional which measures the potential interaction contained in the Cauchy data along any space-like curve. Glimm shows that this functional decreases with time.

The notion of entropy discussed here has been proposed by Lax (1971), and Krushkov (1970). The theory of the symmetric case is due to Godunov and was applied by him to the compressible flow equations.

6. *Hyperbolic systems of two conservation laws.* The nonexistence Theorem 6.1 and Theorem 6.2 are from Lax (1964); another version has been given by Zabusky (1962).

Johnson and Smoller have shown that under assumption (6.18) the Riemann initial value problem can be solved uniquely for two arbitrary initial states, not necessarily close. They have shown how to solve the initial value problem for such systems under a monotonicity assumption for the initial values. Nishida has shown that for the system

$$u_t - v_x = 0,$$

$$v_t + \left(\frac{1}{w}\right)_x = 0,$$

the initial value problem can be solved for arbitrary initial values $u(x), v_0(x)$, $u_0 \geqq 0$. Nishida's work has been extended by Bakhvalov, DiPerna, Greenberg and Nishida and Smoller.

The Riemann initial value problem in gas dynamics for a broad class of equations of state has been studied by Wendroff.

Theorem 6.2 is due to Glimm and Lax. Further uniqueness theorems, in the absence of rarefaction waves have been given by Rozhdestvenskii and by Hurd. The construction of the entropy function (6.24) is carried out in greater detail in Lax (1971).

7. *Difference schemes.* No set of lectures on hyperbolic conservation laws should end without mention of the various effective difference schemes for computing solutions of conservation laws. These are used to compute solutions of specific initial value problems which come up in scientific and technological problems; problems involving two space dimensions can be handled as well. In addition to providing numerical answers to specific questions, one hopes that numerical calculations will reveal patterns which play a role in the theory to be developed about solutions of these equations.

If it were possible to prove rigorously that solutions of finite difference equations converge, this would provide a proof of the existence of solutions with arbitrarily prescribed data. So far this has been accomplished only for single conservation laws, and for a very crude difference scheme proposed by Lax (1954):

$$u_k^{n+1} = \frac{u_{k+1}^n + u_{k-1}^n}{2} + \frac{\Delta t}{2\Delta x}\{f_{k-1}^n - f_{k+1}^n\}. \tag{8}$$

Here u_k^n abbreviates an approximation to u at $t = n\Delta t$, $x = k\Delta x$, and f_k^n abbreviates $f(u_k^n)$. The convergence of this scheme for a special case was verified by Lax (1957); convergence for any convex f was proved by Vvedenskaya. Convergence for any number of space variables was shown by Conway and Smoller, and also by Kotlow.

The approximation (8) is in conservation form: that is, if we think of u_k^n as an approximation to the average value of u over the cell $[(k - \frac{1}{2})\Delta x, (k + \frac{1}{2})\Delta x]$ at time $t - n\Delta t$, (8) is of the generic form

$$u_k^{n+1} = u_k^n + \frac{\Delta t}{\Delta x}\{\tilde{f}_{k-1/2} - \tilde{f}_{k+1/2}\}, \tag{9}$$

i.e., where the average value of u in the kth cell at time $t = (n + 1)\Delta t$ differs from the average at time $n\Delta t$ by the average of the amount that has entered and left at the endpoints during the time elapsed. The conservation character of the approximate equation (9) is expressed by the fact that the amount that enters the kth cell during the time interval $\{n\Delta t, (n + 1)\Delta t\}$ through the left endpoint is exactly equal to the amount which leaves the $(k - 1)$st cell through its right endpoint during the same time interval.

In (8), $\tilde{f}_{k+1/2}$ was taken to be

$$\tilde{f}_{k+1/2} = \frac{f_{k+1}^n + f_k^n}{2} + \frac{\Delta x}{\Delta t}\frac{u_k^n - u_{k+1}^n}{2}; \tag{10}$$

this is a rather poor approximation to the average flux at $k + \frac{1}{2}$ during $\{n, n + 1\}$, since the flux instead of being an average is evaluated at the earliest time n. In addition, for Δt small, the presence of a rather large amount of artificial viscosity proportional to $(\Delta x)^2 u_{xx}/\Delta t$ causes additional errors. This second term has to be included to stabilize (8); to ensure stability one has to impose in addition the CFL condition (see Courant, Friedrichs and Lewy):

$$\frac{\Delta t}{\Delta x} \leqq \frac{1}{\lambda_{\max}}. \tag{11}$$

A more accurate choice of $\tilde{f}_{k+1/2}$ has been proposed by Lax and Wendroff (1960), (1964). Starting with the Taylor series

$$u^{n+1} = u + \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt}$$

and the formulas

$$u_t = -f_x = -Au_x, \qquad u_{tt} = -f_{xt} = -f_{tx} = (-Au_t)_x = (A^2 u_x)_x = (Af_x)_x,$$

set in (9)

$$\tilde{f}_{k+1/2} = \frac{f^n_{k+1} + f^n_k}{2} - \frac{\Delta t}{2\Delta x} A_{k+1/2}(f_{k+1} - f_k). \tag{12}$$

Since this formula centers the flux properly at time $t = (n + \frac{1}{2})\Delta$, it is more accurate than (8); it can be shown that this formula is stable if the CFL condition (11) is satisfied.

The following modification of (12), proposed by Richtmyer (see Richtmyer and Morton), turns out to be more practical:

$$\tilde{f}_{k+1/2} = f(\bar{u}^{n+1/2}_{k+1/2}), \tag{13}$$

where

$$\bar{u}^{n+1/2}_{k+1/2} = \frac{u^n_k + u^n_{k+1}}{2} + \frac{\Delta t}{2\Delta x}\{f^n_k - f^n_{k+1}\}. \tag{14}$$

A further interesting modification which has been introduced by R. W. MacCormack has been especially efficient in the case of several space variables.

Recently, still more accurate schemes have been devised by Rusanov and by Burstein and Mirin.

Another type of difference scheme has been introduced by Godunov; his starting point is the same as in Glimm's scheme, but the approximate average value at time $(n + 1)\Delta t$ over the kth cell is defined to be the average of the exact solution computed there. This average value is computed from the flux relation, i.e., (9) is used, with $\tilde{f}_{k+1/2}$ taken as the exact value of f at the interface between the kth and $(k + 1)$st cell.

Calculations performed with the methods described above produce approximate solutions in which a shock is spread over a finite number—usually two to four—of meshpoints. This is in sharp contrast to numerical solutions or linear equations, where discontinuities are spread over regions which are proportional to some power of the number of time steps taken.

A different computing method, called particle-in-cell, has been developed by Harlow. This method is particularly effective in several space dimensions.

A study of the formation of steady state profiles for solutions of difference equations was begun by Jennings (1973).

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